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Larson–Sweedler theorem and the role of grouplike elements in weak Hopf algebras

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Abstract

We extend the Larson–Sweedler theorem [Amer. J. Math. 91 (1969) 75] to weak Hopf algebras by proving that a finite dimensional weak bialgebra is a weak Hopf algebra iff it possesses a non-degenerate left integral. We show that the category of modules over a weak Hopf algebra is autonomous monoidal with semisimple unit and invertible modules. We also reveal the connection of invertible modules to left and right grouplike elements in the dual weak Hopf algebra. Defining distinguished left and right grouplike elements, we derive the Radford formula [Amer. J. Math. 98 (1976) 333] for the fourth power of the antipode in a weak Hopf algebra and prove that the order of the antipode is finite up to an inner automorphism by a grouplike element in the trivial subalgebra A^T of the underlying weak Hopf algebra A.

0. Introduction

Weak Hopf algebras have been proposed recently [1,2,18] as a generalization of Hopf algebras by weakening the compatibility conditions between the algebra and coalgebra structures of Hopf algebras. Comultiplication is allowed to be non-unital, $\Delta(\mathbf{1}) \equiv \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)} \neq \mathbf{1} \otimes \mathbf{1}$, just like in weak quasi Hopf algebras [11] and in rational Hopf algebras [8,19], but the comultiplication is coassociative. In exchange for coassociativity, the multiplicativity of the counit is replaced by a weaker condition: $\varepsilon(ab) = \varepsilon(a\mathbf{1}^{(1)})\varepsilon(\mathbf{1}^{(2)}b)$, implying that the unit representation is not necessarily one-dimensional and irreducible.

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Like weak quasi and rational Hopf algebras, they can possess non-integral (quantum) dimensions even in the finite dimensional and semisimple cases, which is necessary if we want to recover them as global symmetries of low-dimensional quantum field theories. In situations where only the representation category matters, these two concepts are equivalent. Nevertheless, just like finite dimensional Hopf algebras, finite dimensional weak Hopf algebras (WHA) obey the mathematical beauty of giving rise to a self-dual notion: the dual space of a WHA can be canonically endowed with a WHA structure. For a recent review, see [12].

Here we continue the study [2] of the structural properties of finite dimensional weak Hopf algebras over a field k. The main results of this paper are:

(1) The generalization of the Larson–Sweedler theorem [10] to WHAs, claiming that a finite dimensional weak bialgebra is a weak Hopf algebra if and only if it possesses a non-degenerate left integral.
(2) The characterization of inequivalent invertible modules of WHAs through left/right grouplike elements in the dual WHA and the proof of the semisimplicity of invertible modules, which include the unit module serving as a monoidal unit in the monoidal category of left (right) modules.
(3) A finiteness claim about the order of the antipode (up to an inner automorphism by a grouplike element in the trivial subalgebra) and the derivation of the Radford formula [15] in a weak Hopf algebra A: $S^4(a) = \sigma \rightharpoonup s^{-1}as \leftharpoonup \widehat{S}^{-1}(\sigma),\ a \in A$, where S $(\widehat{S})$ is the antipode in A $(\widehat{A})$, and s and σ are distinguished left grouplike elements in A and in the dual WHA $\widehat{A}$, respectively.

The existence of a non-degenerate left integral $l \in B$ in a finite dimensional bialgebra B implies the existence of a non-degenerate left integral $\lambda \in \widehat{B}$ in the dual bialgebra $\widehat{B}$ with the property $\lambda \rightharpoonup l = \mathbf{1}$. Then the formula $S(a) := (\lambda \leftharpoonup a) \rightharpoonup l$, $a \in B$ gives rise to the antipode for B proving one direction of the Larson–Sweedler theorem [10]. The proof of the opposite direction [10] involves the structure theorem for Hopf modules, which are one-sided H-modules and H-comodules of the Hopf algebra H together with a compatibility condition. The structure theorem for a finite dimensional Hopf module M_H^H claims that $M_H^H \simeq C(M) \otimes H_H^H$ as Hopf modules, where $C(M)$ is the space of coinvariants in M_H^H and H_H^H is the canonical Hopf module. Proving that the dual Hopf algebra $\widehat{H}$ carries a Hopf module structure, $\widehat{H}_H^H \simeq C(\widehat{H}) \otimes H_H^H$ follows and observing that by dimensionality argument $C(\widehat{H})$ is one-dimensional, a non-degenerate left integral in $\widehat{H}$ emerges in the space of coinvariants $C(\widehat{H})$.

The proof of the corresponding statement (Theorem 4.1) in the case of finite dimensional weak bialgebras is in the same spirit. The existence of a non-degenerate left integral in a finite dimensional WBA implies the existence of a non-degenerate left integral in the dual WBA and the previous classical formula leads to the antipode. The proof of the opposite direction is more involved: besides weak Hopf modules one has to introduce multiple weak Hopf modules, in which bimodule or bicomodule structures are also present together with compatibility conditions between the module and comodule structures. Then the structure theorem (Theorem 3.2) for a multiple weak Hopf module ${}_AM_A^A$ of a WHA A claims that ${}_AM_A^A \simeq {}_A(C(M) \times A_A^A)$, i.e., the right weak Hopf module structure of M is given by the canonical weak Hopf module A_A^A; while as a left A-module, M is isomorphic to the product module of the coinvariants ${}_AC(M)$ and the left regular module ${}_AA$. The left A-module structure of the coinvariants arises from the bimodule structure of M (Lemma 3.1(iii)). In particular, the dual WHA $\widehat{A}$ is a multiple weak Hopf module ${}_A\widehat{A}_A^A$ and its coinvariants $C(\widehat{A})$ are the left integrals $\hat{I}^L \subset \widehat{A}$ (Theorem 3.2). Moreover, $\hat{I}^L$ becomes a free left A^R- and A^L-module with a single generator by restricting the left A-module structure of ${}_A\hat{I}^L$ to the canonical coideal subalgebras A^R and A^L of A, respectively (Corollary 3.5). It is the latter result that replaces the dimensionality argument of the classical Hopf case and, together with the isomorphism ${}_A\widehat{A}_A^A \simeq {}_A(\hat{I}^L \times A_A^A)$ of multiple weak Hopf modules, leads to the existence of a non-degenerate left integral in $\hat{I}^L \subset \widehat{A}$.

The modules of a WHA that are invertible with respect to their monoidal product are important in low dimensional quantum field theories. Hence, it is worth characterising them in purely (weak) Hopf algebraic terms. Although a WHA A is not a semisimple algebra in general, its unit and invertible modules are semisimple (Theorem 2.4, respectively Proposition 5.4(ii)). The origin of this property is that the trivial subWHA A^T, which is generated by the canonical coideal subalgebras A^L and A^R of a WHA A, is in the coradical of A (Lemma 2.3). We derive two other equivalent characterizations of invertible modules: they are precisely the modules that become free rank one A^L- and A^R-modules by restricting the A-module structure to these coideal subalgebras (Proposition 5.4(i)). For example, the invertible left A-module structure of right integrals $I^R \subset A$ and left integrals $\hat{I}^L \subset \widehat{A}$ follows in this way. The second equivalent characterization of invertible A-modules involves left or right grouplike elements (Definition 5.1) in the dual WHA: an A-module is invertible iff it is isomorphic to a cyclic submodule in the second regular A-module ${}_A\widehat{A}$ generated by a left (right) grouplike element in $\widehat{A}$ (Proposition 5.7). Moreover, the isomorphism classes of invertible A-modules are given by the (finite) factor group $G_L(\widehat{A})/G_L^T(\widehat{A})$ (or by $G_R(\widehat{A})/G_R^T(\widehat{A})$) (Proposition 5.7), where $G_L^T(\widehat{A})$ is the intersection of the (in general infinite) set of left grouplike elements $G_L(\widehat{A})$ and the trivial subWHA $\widehat{A}^T$ in $\widehat{A}$.

If $l \in A$ and $\lambda \in \widehat{A}$ are dual left integrals, i.e., if they are non-degenerate and satisfy $\lambda \rightharpoonup l = \mathbf{1}$, then $s := l \leftharpoonup \lambda$ and $\sigma := \lambda \leftharpoonup l$ will define (distinguished) left grouplike elements (Definition 6.1 and discussion before) like in the Hopf case [15]. σ falls into a central element of the factor group $G_L(\widehat{A})/G_L^T(\widehat{A})$ and determines the unimodularity of A, that is the possible existence of a two-sided non-degenerate integral in A (Corollary 6.3). The Nakayama automorphism $\theta_\lambda : A \to A$ corresponding to a non-degenerate left integral $\lambda \in \widehat{A}$ can be given in terms of distinguished left grouplike elements in two different ways, which contain the square or the inverse square of the antipode. Hence, these expressions lead to the generalization of the Radford formula [15] to WHAs (Theorem 6.4). Since the factor groups $G_L(A)/G_L^T(A)$ and $G_L(\widehat{A})/G_L^T(\widehat{A})$ are finite and since even powers of the antipode are WHA automorphisms, the iteration of the Radford formula leads to the claim that the order of the antipode is finite up to a conjugation by an element in $G_L^T(A) \cap G_R^T(A)$ (Theorem 6.4). The explicit form of the Nakayama automorphism θ_λ, like in the Hopf case [16], can be used to prove the unimodularity of the double of a WHA (Corollary 6.5).

We note that it was established in [2] that WHAs are quasi-Frobenius algebras. Result 1 implies that they are Frobenius algebras. Grouplike elements in a WHA, which are just the intersection of left and right grouplike elements in our formulation, were introduced in [2]. The modules associated with them were studied in [13]. However, this notion of grouplike elements is too restrictive: for characterization of isomorphism classes of invertible modules (Result 2) one has to introduce the less restrictive notion of left (right) grouplike elements, because the factor group $G(\widehat{A})/G^T(\widehat{A})$ of grouplike and trivial grouplike elements is, in general, smaller than the corresponding factor group $G_L(A)/G_L^T(A)$ of left grouplike elements (Proposition 5.8). Result 3 was proved in [13] in the case when the square of the antipode is the identity mapping on the coideal subalgebra A^L of the WHA A.

The organization of the paper is as follows. In Section 1 we review the axioms and the main properties of weak bialgebras (WBA) and weak Hopf algebras. Here and throughout the paper they are considered to be finite dimensional. Section 2 is devoted to the autonomous monoidal category of modules of a WHA and to properties of the unit module including semisimplicity. We derive also a lower bound for the k-dimension of an A-module in terms of the k-dimensions of the simple submodules of the unit A-module. This estimation leads to a sufficient condition for an A-module to become a free rank one A^L- and A^R-module. In Section 3 we prove a structure theorem for multiple weak Hopf modules and show that the left A-modules spanned by right integrals in A and left integrals in $\widehat{A}$ become free rank one A^L- and A^R-modules. Section 4 contains the generalization of the Larson–Sweedler theorem to the weak Hopf case. In Section 5 we reveal the connection between invertible modules of a WHA A and left (right) grouplike elements in the dual WHA $\widehat{A}$ and prove that invertible modules are semisimple. Section 6 contains the definition and some basic properties of distinguished left and right grouplike elements, the derivations of the form of the Nakayama automorphism $\theta_\lambda : A \to A$ corresponding to a non-degenerate left integral $\lambda \in \widehat{A}$ and the Radford formula. In addition, we prove the claim about the order of the antipode and unimodularity of the double of a WHA. In Appendix A we give a simple example of a WHA in which the order of the antipode is not finite. Finally, Appendix B contains the generalization of the cyclic category module [4] to weak Hopf algebras containing a modular pair of grouplike elements in involution.

1. Preliminaries

Here we give a quick survey of weak bialgebras and weak Hopf algebras [2]. We restrict ourselves to their main properties, however, some useful identities we use later on are also given.

1.1. The axioms

A *weak bialgebra* $(A; u, \mu; \varepsilon, \Delta)$ is defined by the properties (i)–(iii):

(i) A is a finite dimensional associative algebra over a field k with multiplication $\mu : A \otimes A \to A$ and unit $u : k \to A$, which are k-linear maps.
(ii) A is a coalgebra over k with comultiplication $\Delta : A \to A \otimes A$ and counit $\varepsilon : A \to k$, which are k-linear maps.

(iii) The algebra and coalgebra structures obey the compatibility conditions

$$\Delta(ab) = \Delta(a)\Delta(b), \quad a, b \in A, \tag{1.1a}$$

$$\varepsilon\big(ab^{(1)}\big)\varepsilon\big(b^{(2)}c\big) = \varepsilon(abc) = \varepsilon\big(ab^{(2)}\big)\varepsilon\big(b^{(1)}c\big), \quad a, b, c \in A, \tag{1.1b}$$

$$\mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)}\mathbf{1}^{(1')} \otimes \mathbf{1}^{(2')} = \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)} \otimes \mathbf{1}^{(3)} = \mathbf{1}^{(1)} \otimes \mathbf{1}^{(1')}\mathbf{1}^{(2)} \otimes \mathbf{1}^{(2')}, \tag{1.1c}$$

where (and later on) $ab \equiv \mu(a, b)$, $\mathbf{1} := u(1)$ and we used Sweedler notation [17] for iterated coproducts omitting summation indices and a summation symbol.

A *weak Hopf algebra* $(A; u, \mu; \varepsilon, \Delta; S)$ is a WBA together with property (iv):

(iv) There exists a k-linear map $S : A \to A$, called the antipode, satisfying

$$a^{(1)}S\big(a^{(2)}\big) = \varepsilon\big(\mathbf{1}^{(1)}a\big)\mathbf{1}^{(2)}, \tag{1.2a}$$

$$S\big(a^{(1)}\big)a^{(2)} = \mathbf{1}^{(1)}\varepsilon\big(a\mathbf{1}^{(2)}\big), \quad a \in A, \tag{1.2b}$$

$$S\big(a^{(1)}\big)a^{(2)}S\big(a^{(3)}\big) = S(a). \tag{1.2c}$$

WBAs and WHAs are self-dual notions, the dual space $\widehat{A} := \mathrm{Hom}_k(A, k)$ of a WBA (WHA) equipped with structure maps $\hat{u}, \hat{\mu}, \hat{\varepsilon}, \widehat{\Delta}, (\widehat{S})$ defined by transposing the structure maps of A by means of the canonical pairing $\langle , \rangle : \widehat{A} \times A \to k$ gives rise to a WBA (WHA).

1.2. Properties of WBAs

Let A be a WBA. The images $A^{L/R} = \Pi^{L/R}(A) = \overline{\Pi}^{L/R}(A)$ of the projections $\Pi^{L/R} : A \to A$ and $\overline{\Pi}^{L/R} : A \to A$ defined by

$$\begin{aligned} &\Pi^L(a) := \varepsilon\big(\mathbf{1}^{(1)}a\big)\mathbf{1}^{(2)}, \qquad \Pi^R(a) := \mathbf{1}^{(1)}\varepsilon\big(a\mathbf{1}^{(2)}\big), \\ &\overline{\Pi}^L(a) := \varepsilon\big(a\mathbf{1}^{(1)}\big)\mathbf{1}^{(2)}, \qquad \overline{\Pi}^R(a) := \mathbf{1}^{(1)}\varepsilon\big(\mathbf{1}^{(2)}a\big), \quad a \in A \end{aligned} \tag{1.3}$$

are unital subalgebras (i.e., containing $\mathbf{1}$) of A that commute with each other. A^L and A^R are called *left* and *right subalgebras*, respectively. The image $\Delta(\mathbf{1})$ of the unit is in $A^R \otimes A^L$ and the coproduct on $A^{L/R}$ reads as

$$\Delta\big(x^L\big) = \mathbf{1}^{(1)}x^L \otimes \mathbf{1}^{(2)}, \quad x^L \in A^L, \qquad \Delta\big(x^R\big) = \mathbf{1}^{(1)} \otimes x^R\mathbf{1}^{(2)}, \quad x^R \in A^R. \tag{1.4}$$

Hence, A^L and A^R are left and right coideals, respectively, and the *trivial subalgebra* $A^T := A^L \vee A^R \subset A$ generated by the coideal subalgebras A^L and A^R is a subWBA of A.

The maps $\kappa_L : A^L \to \widehat{A}^R$ and $\kappa_R : A^R \to \widehat{A}^L$ given by Sweedler arrows

$$\kappa_L\big(x^L\big) := x^L \rightharpoonup \hat{\mathbf{1}}, \qquad \kappa_R\big(x^R\big) := \hat{\mathbf{1}} \leftharpoonup x^R, \quad x^{L/R} \in A^{L/R} \tag{1.5}$$

are algebra isomorphisms with inverses $\hat{\kappa}_R$ and $\hat{\kappa}_L$, respectively. Moreover,

$$x^L \rightharpoonup \varphi = \bigl(x^L \rightharpoonup \hat{\mathbf{1}}\bigr)\varphi, \qquad \varphi \leftharpoonup x^L = \bigl(\hat{\mathbf{1}} \leftharpoonup x^L\bigr)\varphi,$$
$$x^R \rightharpoonup \varphi = \varphi\bigl(x^R \rightharpoonup \hat{\mathbf{1}}\bigr), \qquad \varphi \leftharpoonup x^R = \varphi\bigl(\hat{\mathbf{1}} \leftharpoonup x^R\bigr), \quad \varphi \in \widehat{A},\ x^{L/R} \in A^{L/R}. \tag{1.6}$$

Defining $Z^{L/R} := A^{L/R} \cap \text{Center}\, A$ and $Z := A^L \cap A^R$ the restrictions of $\kappa_{L/R}$ to $Z^{L/R}$ and Z lead to the algebra isomorphisms $Z^{L/R} \to \widehat{Z}$ and $Z \to \widehat{Z}^{R/L}$, respectively. Hence, the *hypercenter* $H := Z \cap \text{Center}\, A = Z^L \cap Z = Z^R \cap Z$ of A is isomorphic to the hypercenter $\widehat{H}$ of $\widehat{A}$ via the restriction of κ_L or κ_R to H.

The restrictions of the canonical pairing to $\widehat{A}^{L/R} \times A^{L/R}$ (four possibilities) are non-degenerate. The maps $\Pi^{L/R}$ and $\widehat{\Pi}^{L/R}$ ($\overline{\Pi}^{L/R}$ and $\widehat{\overline{\Pi}}^{R/L}$) are transposed to each other:

$$\bigl\langle \varphi, \Pi^{L/R}(a)\bigr\rangle = \bigl\langle \widehat{\Pi}^{L/R}(\varphi), a\bigr\rangle,$$
$$\bigl\langle \varphi, \overline{\Pi}^{L/R}(a)\bigr\rangle = \bigl\langle \widehat{\overline{\Pi}}^{R/L}(\varphi), a\bigr\rangle, \quad a \in A, \varphi \in \widehat{A} \tag{1.7}$$

(note the switch of L and R in the second equation) and obey the identities

$$\Pi^R\bigl(a^{(1)}\bigr) \otimes a^{(2)} = \mathbf{1}^{(1)} \otimes a\mathbf{1}^{(2)}, \qquad \overline{\Pi}^R\bigl(a^{(1)}\bigr) \otimes a^{(2)} = \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)}a,$$
$$a^{(1)} \otimes \Pi^L\bigl(a^{(2)}\bigr) = \mathbf{1}^{(1)}a \otimes \mathbf{1}^{(2)}, \qquad a^{(1)} \otimes \overline{\Pi}^L\bigl(a^{(2)}\bigr) = a\mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)}, \quad a \in A \tag{1.8}$$

due to (1.1c). The space of *left/right integrals* $I^{L/R}$ in A is defined by

$$I^L := \bigl\{l \in A \bigm| al = \Pi^L(a)l,\ a \in A\bigr\}, \qquad I^R := \bigl\{r \in A \bigm| ra = r\Pi^R(a),\ a \in A\bigr\}. \tag{1.9}$$

1.3. Properties of WHAs

Let A be a WHA. The antipode S, as in the case of Hopf algebras, turns out to be invertible, antimultiplicative, anticomultiplicative and leaves the counit invariant $\varepsilon = \varepsilon \circ S$. The restriction of the antipode to A^L leads to algebra antiisomorphism $S : A^L \to A^R$, therefore A^T is a subWHA of A, moreover,

$$\Pi^L \circ S = \Pi^L \circ \Pi^R = S \circ \Pi^R, \qquad \Pi^R \circ S = \Pi^R \circ \Pi^L = S \circ \Pi^L. \tag{1.10}$$

The projections (1.3) to left and right subalgebras can be expressed as

$$\Pi^L(a) = a^{(1)}S\bigl(a^{(2)}\bigr), \qquad \Pi^R(a) = S\bigl(a^{(1)}\bigr)a^{(2)},$$
$$\overline{\Pi}^L(a) = S^{-1}\bigl(\Pi^R(a)\bigr), \qquad \overline{\Pi}^R(a) = S^{-1}\bigl(\Pi^L(a)\bigr), \quad a \in A. \tag{1.11}$$

The first two equations follow from the antipode axioms (1.2a and b). The other two can be seen using the aforementioned properties of the antipode and the WBA identity $\varepsilon(abc) = \varepsilon(\Pi^R(a)b\Pi^L(c))$ following from (1.1b) and (1.3). The left and right subalgebras become separable k-algebras with separating idempotents [14, p. 182] $q^L = S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} \in A^L \otimes A^L$ and $q^R = \mathbf{1}^{(1)} \otimes S(\mathbf{1}^{(2)}) \in A^R \otimes A^R$, respectively, that obey

$$x^L S\big(\mathbf{1}^{(1)}\big) \otimes \mathbf{1}^{(2)} = S\big(\mathbf{1}^{(1)}\big) \otimes \mathbf{1}^{(2)} x^L, \quad x^L \in A^L,$$
$$x^R \mathbf{1}^{(1)} \otimes S\big(\mathbf{1}^{(2)}\big) = \mathbf{1}^{(1)} \otimes S\big(\mathbf{1}^{(2)}\big) x^R, \quad x^R \in A^R \tag{1.12}$$

by definition. The product $q^L q^R \in A^T \otimes A^T$ is a separating idempotent for A^T, thus the trivial subalgebra is a separable k-algebra, too. The separating idempotent $q^{L/R}$ serves as a *quasibasis* [20, p. 6] for the counit

$$S\big(\mathbf{1}^{(1)}\big)\varepsilon\big(\mathbf{1}^{(2)} x^L\big) = x^L = \varepsilon\big(x^L S\big(\mathbf{1}^{(1)}\big)\big)\mathbf{1}^{(2)}, \quad x^L \in A^L,$$
$$\mathbf{1}^{(1)}\varepsilon\big(S\big(\mathbf{1}^{(2)}\big)x^R\big) = x^R = \varepsilon\big(x^R \mathbf{1}^{(1)}\big) S\big(\mathbf{1}^{(2)}\big), \quad x^R \in A^R, \tag{1.13}$$

thus the counit is a non-degenerate functional on $A^{L/R}$. The properties $S(\mathbf{1}^{(1)})\mathbf{1}^{(2)} = \mathbf{1}$ and $\mathbf{1}^{(1)} S(\mathbf{1}^{(2)}) = \mathbf{1}$ of separating idempotents q_L and q_R ensure that the counit ε is an index $\mathbf{1}$ functional [20, p. 7] on A^L and on A^R, respectively. Due to the identities (1.5), (1.7), (1.10), and (1.12), the corresponding Nakayama automorphisms $\theta_{L/R} : A^{L/R} \to A^{L/R}$, which are defined by

$$\varepsilon\big(y^{L/R}\theta_{L/R}\big(x^{L/R}\big)\big) := \varepsilon\big(x^{L/R} y^{L/R}\big), \quad x^{L/R}, y^{L/R} \in A^{L/R}, \tag{1.14}$$

can be given as

$$\theta_L\big(x^L\big) = \mathbf{1} \leftharpoonup \widehat{S}^{-1}\big(\hat{\mathbf{1}} \leftharpoonup x^L\big) = S^2\big(x^L\big), \quad x^L \in A^L,$$
$$\theta_R\big(x^R\big) = \widehat{S}\big(\hat{\mathbf{1}} \leftharpoonup x^R\big) \rightharpoonup \mathbf{1} = S^{-2}\big(x^R\big), \quad x^R \in A^R. \tag{1.15}$$

Hence, θ_L (θ_R) is the restriction of the square of the (inverse of the) antipode to A^L (A^R). Since any separable algebra admits a non-degenerate (reduced) trace [6, p. 165], the counit, being a non-degenerate functional on $A^{L/R}$, can be given with the help of the corresponding trace as $\varepsilon(\cdot) = \mathrm{tr}_{L/R}(t_{L/R}\cdot)$ with $t_{L/R} \in A^{L/R}$ invertible. Therefore, the Nakayama automorphisms $\theta_{L/R}$ are given by $\mathrm{ad}\, t_{L/R}$ and S^2 is inner on $A^{L/R}$, hence, on A^T, too.

In a WHA a left integral $l \in I^L$ and a right integral $r \in I^R$ obey the identities

$$l^{(1)} \otimes a l^{(2)} = S(a) l^{(1)} \otimes l^{(2)}, \qquad r^{(1)} a \otimes r^{(2)} = r^{(1)} \otimes r^{(2)} S(a), \quad a \in A, \tag{1.16}$$

respectively. Moreover, there exist projections $L/R : A \to I^{L/R}$ and $\overline{L}/\overline{R} : A \to I^{L/R}$:

$$L(a) := \widehat{S}^2(\beta_i) \rightharpoonup (b_i a), \qquad R(a) := (a b_i) \leftharpoonup \widehat{S}^2(\beta_i),$$
$$\overline{L}(a) := (b_i a) \leftharpoonup \widehat{S}^{-2}(\beta_i), \qquad \overline{R}(a) := \widehat{S}^{-2}(\beta_i) \rightharpoonup (a b_i), \quad a \in A \tag{1.17}$$

where $\{b_i\} \subset A$ and $\{\beta_i\} \subset \widehat{A}$ are dual k-bases with respect to the canonical pairing. They obey the properties

$$\big\langle \widehat{L}/\widehat{R}(\varphi), a\big\rangle = \big\langle \varphi, R/L(a)\big\rangle, \qquad \big\langle \widehat{\overline{L}}/\widehat{\overline{R}}(\varphi), a\big\rangle = \big\langle \varphi, \overline{L}/\overline{R}(a)\big\rangle, \quad a \in A,\ \varphi \in \widehat{A}, \tag{1.18}$$

therefore, the restrictions of the canonical pairing to $\hat{I}^{L/R} \times I^{L/R}$ (four possibilities) are non-degenerate.

2. Properties of the unit module

In this chapter A denotes a WHA over a field k.

A *left* (*right*) *A-module* ${}_AM \equiv (M, \mu_L)$ $(M_A \equiv (M, \mu_R))$ is a k-linear space together with the k-linear map $\mu_L : A \otimes M \to M$ $(\mu_R : M \otimes A \to M)$ satisfying

$$\begin{aligned} a \cdot (b \cdot m) &= (ab) \cdot m, & (m \cdot a) \cdot b &= m \cdot (ab), \\ \mathbf{1} \cdot m &= m, & m \cdot \mathbf{1} &= m, \qquad m \in M,\ a, b \in A, \end{aligned}$$

where (and later on) $\mu_L(a \otimes m) \equiv a \cdot m$ and $\mu_R(m \otimes a) \equiv m \cdot a$. The role of the unit module will be played by the trivial representation [2, p. 400] of A.

Definition 2.1. The unit left (right) A-module ${}_AA^L$ (A^R_A) is defined by

$$\begin{aligned} a \cdot x^L &:= \Pi^L\big(ax^L\big) = a^{(1)} x^L S\big(a^{(2)}\big), \quad x^L \in A^L,\ a \in A, \\ x^R \cdot a &:= \Pi^R\big(x^R a\big) = S\big(a^{(1)}\big) x^R a^{(2)}, \quad x^R \in A^R,\ a \in A. \end{aligned} \tag{2.1}$$

We note that these modules need not be one-dimensional as in the case of Hopf algebras, they are not even simple in general. Nevertheless, they play the role of the unit object in the monoidal category of finite dimensional left (right) A-modules. We deal with only the category of left A-modules since the one-to-one correspondence between left and right A-modules induced by the antipode, $m \cdot a := S(a) \cdot m$, $a \in A$, $m \in {}_AM$, extends to a categorical isomorphism.

Proposition 2.2. *The category $\mathcal{L}$, consisting of finite dimensional left A-modules of a WHA A as objects and left A-module maps as arrows, can be endowed with an autonomous* (*relaxed*) *monoidal structure*

$$\big(\mathcal{L}; \times, {}_AA^L, \{1_{K\times M\times N}\}, \big\{X^L_M\big\}, \big\{X^R_M\big\}; \overset{\leftharpoonup}{\ }, \overset{\rightharpoonup}{\ }\big),$$

where $\times$ is the monoidal product, ${}_AA^L$ is the monoidal unit, $\{1_{K\times M\times N}\}$, $\big\{X^L_M\big\}$, $\big\{X^R_M\big\}$ are natural equivalences satisfying the pentagon and the triangle identities, while $\overset{\leftharpoonup}{\ }$ and $\overset{\rightharpoonup}{\ }$ are the functors of left and right conjugations, respectively.

Proof. Let us define first the monoidal product $\times$. The product module ${}_A(M \times N)$ of the modules ${}_AM$ and ${}_AN$ as a k-linear space is defined to be

$$M \times N := \mathbf{1}^{(1)} \cdot M \otimes \mathbf{1}^{(2)} \cdot N \tag{2.2a}$$

and the left A-module structure on $M \times N$ is given by

$$a \cdot (m \otimes n) := a^{(1)} \cdot m \otimes a^{(2)} \cdot n, \quad a \in A,\ m \otimes n \in M \times N, \tag{2.2b}$$

where (and later on) we have suppressed possible or necessary summation for tensor product elements in product modules. The product on the arrows $T_\alpha : M_\alpha \to N_\alpha$, $\alpha = 1, 2$ is defined by $T_1 \times T_2 := (T_1 \otimes T_2) \circ \Delta(\mathbf{1})$, i.e., by the restriction of the tensor product of the linear maps T_1 and T_2 to $M_1 \times M_2$. One can easily check that $T_1 \times T_2 : M_1 \times M_2 \to N_1 \times N_2$ is a left A-module map. The given monoidal product is associative due to the associativity of the coproduct and property (1.1c) of the unit, hence the components $M_1 \times (M_2 \times M_3) \to (M_1 \times M_2) \times M_3$ of the natural equivalence responsible for associativity in a monoidal category are the identity mappings $1_{M_1 \times M_2 \times M_3}$ in our case.

The monoidal unit property of the left A-module A^L can be seen by verifying that for any object M the k-linear invertible maps $X^L_M : M \to A^L \times M$ and $X^R_M : M \to M \times A^L$ defined by

$$\begin{aligned} X^L_M(m) &:= S\big(\mathbf{1}^{(1)}\big) \otimes \mathbf{1}^{(2)} \cdot m, & X^R_M(m) &:= \mathbf{1}^{(1)} \cdot m \otimes \mathbf{1}^{(2)}, \\ \big(X^L_M\big)^{-1}\big(x^L \otimes m\big) &:= x^L \cdot m, & \big(X^R_M\big)^{-1}\big(m \otimes x^L\big) &:= S^{-1}\big(x^L\big) \cdot m, \end{aligned} \tag{2.3}$$

are left A-module maps and the identities

$$\begin{aligned} X^L_N T &= \big(1_{A^L} \times T\big) X^L_M, \\ X^R_N T &= \big(T \times 1_{A^L}\big) X^R_M, \quad M, N \in \operatorname{Obj} \mathcal{L},\ T : M \to N, \end{aligned} \tag{2.4}$$

$$\big(X^R_M \times 1_N\big)\big(1_M \times \big(X^L_N\big)^{-1}\big) = 1_{M \times A^L \times N} \tag{2.5}$$

hold, i.e., $X^L = \{X^L_M\}$ and $X^R = \{X^R_M\}$ are natural equivalences satisfying the triangle identity.

An autonomous category [21] contains both left and right conjugation functors by definition. The left conjugate $\overleftarrow{M}$ of an object M in $\mathcal{L}$ is given by the k-dual $\widehat{M} := \operatorname{Hom}_k(M, k)$ as a k-linear space. The left A-module structure $\overleftarrow{M} \equiv (\widehat{M}, \overleftarrow{\mu}_L)$ is defined to be

$$\langle a \cdot \widehat{m}, m \rangle := \big\langle \widehat{m}, S(a) \cdot m \big\rangle, \quad a \in A,\ m \in M,\ \widehat{m} \in \widehat{M}, \tag{2.6}$$

where $\langle , \rangle$ is the k-valued canonical bilinear paring on the cartesian product of $\widehat{M}$ and M. Dual bases with respect to this pairing will be denoted by $\{\widehat{m}_i\}_i \subset \widehat{M}$ and $\{m_i\}_i \subset M$. Due to the definition (2.6) of the left A-module $\overleftarrow{M}$ we have

$$m_i \otimes a \cdot \widehat{m}_i = S(a) \cdot m_i \otimes \widehat{m}_i, \quad a \in A, \tag{2.7a}$$

$$\begin{aligned} m_i \otimes \widehat{m}_i &= \mathbf{1} \cdot m_i \otimes \widehat{m}_i = \mathbf{1}^{(1)} S\big(\mathbf{1}^{(2)}\big) \cdot m_i \otimes \widehat{m}_i \\ &= \mathbf{1}^{(1)} \cdot m_i \otimes \mathbf{1}^{(2)} \cdot \widehat{m}_i \in M \times \overleftarrow{M}, \end{aligned} \tag{2.7b}$$

where (and later on) we omit summation symbol for the sum of tensor product of dual basis elements.

The arrow family of left evaluation and coevaluation maps $E^l_M : \overleftarrow{M} \times M \to A^L$ and $C^l_M : A^L \to M \times \overleftarrow{M}$, respectively, are defined to be

$$\begin{aligned} E^l_M(\widehat{m} \otimes m) &:= \mathbf{1}^{(2)}\langle \widehat{m}, \mathbf{1}^{(1)} \cdot m\rangle, \quad \widehat{m} \otimes m \in \overleftarrow{M} \times M, \\ C^l_M\bigl(x^L\bigr) &:= x^L \cdot m_i \otimes \widehat{m}_i, \quad x^L \in A^L. \end{aligned} \tag{2.8}$$

They are left A-module maps

$$\begin{aligned} E^l_M\bigl(a \cdot (\widehat{m} \otimes m)\bigr) &= \mathbf{1}^{(2)}\bigl\langle a^{(1)} \cdot \widehat{m}, \mathbf{1}^{(1)} a^{(2)} \cdot m\bigr\rangle = \mathbf{1}^{(2)}\bigl\langle \widehat{m}, S\bigl(a^{(1)}\bigr)\mathbf{1}^{(1)} a^{(2)} \cdot m\bigr\rangle \\ &= \Pi^L\bigl(a^{(3)}\bigr)\bigl\langle \widehat{m}, S\bigl(a^{(1)}\bigr)a^{(2)} \cdot m\bigr\rangle = \Pi^L\bigl(a^{(2)}\bigr)\bigl\langle \widehat{m}, \Pi^R\bigl(a^{(1)}\bigr) \cdot m\bigr\rangle \\ &= \Pi^L\bigl(a\mathbf{1}^{(2)}\bigr)\bigl\langle \widehat{m}, \mathbf{1}^{(1)} \cdot m\bigr\rangle = a \cdot E^l_M(\widehat{m} \otimes m), \\ C^l_M\bigl(a \cdot x^L\bigr) &= a^{(1)} x^L S\bigl(a^{(2)}\bigr) \cdot m_i \otimes \widehat{m}_i = a^{(1)} x^L \cdot m_i \otimes a^{(2)} \cdot \widehat{m}_i \\ &= a \cdot C^l_M\bigl(x^L\bigr) \end{aligned} \tag{2.9}$$

due to the identities (1.8) and (2.7a) and they satisfy the left rigidity identities [21]

$$\begin{aligned} &\bigl(X^R_M\bigr)^{-1}\bigl(1_M \times E^l_M\bigr)\bigl(C^l_M \times 1_M\bigr)X^L_M(m) \\ &\quad := S^{-1}\bigl(\mathbf{1}^{(2')}\bigr)S\bigl(\mathbf{1}^{(1)}\bigr) \cdot m_i\bigl\langle \widehat{m}_i, \mathbf{1}^{(1')}\mathbf{1}^{(2)} \cdot m\bigr\rangle \\ &\quad = S^{-1}\bigl(\mathbf{1}^{(2')}\bigr)S\bigl(\mathbf{1}^{(1)}\bigr)\mathbf{1}^{(1')}\mathbf{1}^{(2)} \cdot m = m, \quad m \in M, \end{aligned} \tag{2.10a}$$

$$\begin{aligned} &\bigl(X^L_{\overleftarrow{M}}\bigr)^{-1}\bigl(E^l_M \times 1_{\overleftarrow{M}}\bigr)\bigl(1_{\overleftarrow{M}} \times C^l_M\bigr)X^R_{\overleftarrow{M}}(\widehat{m}) \\ &\quad := \bigl\langle \mathbf{1}^{(1)} \cdot \widehat{m}, \mathbf{1}^{(1')}\mathbf{1}^{(2)} \cdot m_i\bigr\rangle \mathbf{1}^{(2')} \cdot \widehat{m}_i \\ &\quad = \mathbf{1}^{(2')} S^{-1}\bigl(\mathbf{1}^{(1')}\mathbf{1}^{(2)}\bigr)\mathbf{1}^{(1)} \cdot \widehat{m} = \widehat{m}, \quad \widehat{m} \in \overleftarrow{M} \end{aligned} \tag{2.10b}$$

for any $M \in \operatorname{Obj}\mathcal{L}$. Thus defining the left conjugated arrow $\overleftarrow{T} : \overleftarrow{N} \to \overleftarrow{M}$ of $T : M \to N$ by

$$\overleftarrow{T} := \bigl(X^L_{\overleftarrow{M}}\bigr)^{-1}\bigl(E^l_N \times 1_{\overleftarrow{M}}\bigr)\bigl(1_{\overleftarrow{N}} \times T \times 1_{\overleftarrow{M}}\bigr)\bigl(1_{\overleftarrow{N}} \times C^l_M\bigr)X^R_{\overleftarrow{N}} \tag{2.11}$$

one arrives at the antimonoidal contravariant left conjugation functor $\overleftarrow{\ } : \mathcal{L} \to \mathcal{L}$ [21].

Similarly, the right conjugate $\overrightarrow{M}$ of an object M in $\mathcal{L}$ is the k-linear space $\widehat{M}$ equipped with the left A-module structure $\overrightarrow{M} \equiv (\widehat{M}, \overrightarrow{\mu}_L)$

$$\langle a \cdot \widehat{m}, m\rangle := \bigl\langle \widehat{m}, S^{-1}(a) \cdot m\bigr\rangle, \quad a \in A,\ m \in M,\ \widehat{m} \in \widehat{M} \tag{2.12}$$

implying

$$\widehat{m}_i \otimes a \cdot m_i = S(a) \cdot \widehat{m}_i \otimes m_i, \quad a \in A, \tag{2.13a}$$

$$\begin{aligned} \widehat{m}_i \otimes m_i &= \mathbf{1} \cdot \widehat{m}_i \otimes m_i = \mathbf{1}^{(1)} S\big(\mathbf{1}^{(2)}\big) \cdot \widehat{m}_i \otimes m_i \\ &= \mathbf{1}^{(1)} \cdot \widehat{m}_i \otimes \mathbf{1}^{(2)} \cdot m_i \in \overrightarrow{M} \times M. \end{aligned} \tag{2.13b}$$

The arrow family of right evaluation and coevaluation maps $E^r_M : M \times \overrightarrow{M} \to A^L$ and $C^r_M : A^L \to \overrightarrow{M} \times M$, respectively, are defined to be

$$\begin{aligned} E^r_M(m \otimes \widehat{m}) &:= \mathbf{1}^{(2)} \big\langle \mathbf{1}^{(1)} \cdot \widehat{m}, m \big\rangle, \quad m \otimes \widehat{m} \in M \times \overrightarrow{M}, \\ C^r_M\big(x^L\big) &:= x^L \cdot \widehat{m}_i \otimes m_i, \quad x^L \in A^L. \end{aligned} \tag{2.14}$$

As in the previous case, one proves that they are left A-module maps satisfying the right rigidity identities [21]

$$\begin{aligned} &\big(X^L_M\big)^{-1} \big(E^r_M \times 1_M\big)\big(1_M \times C^r_M\big) X^R_M = 1_M, \\ &\big(X^R_{\overrightarrow{M}}\big)^{-1} \big(1_{\overrightarrow{M}} \times E^r_M\big)\big(C^r_M \times 1_{\overrightarrow{M}}\big) X^L_{\overrightarrow{M}} = 1_{\overrightarrow{M}}. \end{aligned} \tag{2.15}$$

Hence, defining the right conjugated arrow $\overrightarrow{T} : \overrightarrow{N} \to \overrightarrow{M}$ of $T : M \to N$ by

$$\overrightarrow{T} := \big(X^R_{\overrightarrow{M}}\big)^{-1} \big(1_{\overrightarrow{M}} \times E^r_N\big)\big(1_{\overrightarrow{M}} \times T \times 1_{\overrightarrow{N}}\big)\big(C^r_M \times 1_{\overrightarrow{N}}\big) X^L_{\overrightarrow{N}}, \tag{2.16}$$

one arrives at the antimonoidal contravariant right conjugation functor $\overrightarrow{\ } : \mathcal{L} \to \mathcal{L}$. □

In order to prove semisimplicity of the unit module, we show that the trivial subWHA is not only semisimple but also cosemisimple.

Lemma 2.3. *The trivial weak Hopf subalgebra $A^T \subset A$ is a sum of simple subcoalgebras, i.e., A^T is contained in the coradical C_0 of A.*

Proof. First, we decompose the WHA A^T into a direct sum of subWHAs.

The intersection $Z := A^L \cap A^R$ is in the center of the separable algebra A^T, because the unital coideal subalgebras A^L and A^R that generate A^T commute with each other. The WHA identity (1.10) implies $z = S(z)$ for all $z \in Z$. Hence, Z is a unital, pointwise S-invariant subalgebra of the k-algebra $\mathrm{Center}\, A^T$ and one can write A^T as a tensor product algebra $A^T \simeq A^L \otimes_Z A^R$. Let $\{z_\alpha\}_\alpha$ be the set of primitive orthogonal idempotents in Z. They are central idempotents in A^T; thus, $A^T = \bigoplus_\alpha A^T_\alpha$, $A^T_\alpha := A^T z_\alpha$ is an ideal decomposition of the algebra A^T. It is also a WHA decomposition: first, $S(A^T_\alpha) = A^T_\alpha$ since Z is pointwise S-invariant, and second, $\Delta(A^T_\alpha) \subset A^T_\alpha \otimes A^T_\alpha$, because

$$\Delta(x z_\alpha) = \Delta(x z_\alpha z_\alpha) = \Delta(x)(z_\alpha \otimes \mathbf{1})(\mathbf{1} \otimes z_\alpha) = \Delta(x)(z_\alpha \otimes z_\alpha), \quad x \in A^T, \tag{2.17}$$

due to $z_\alpha \in A^L \cap A^R$ and due to coproduct property (1.4) of elements in A^L and in A^R.

This WHA decomposition implies that $(A^T_\alpha)^X = A^X_\alpha$ with $X = L, R, T$ and that the WHA A^T_α has the tensor product algebra structure $A^T_\alpha \simeq A^L_\alpha \otimes_{Z_\alpha} A^R_\alpha$ with unit $\mathbf{1}_\alpha = z_\alpha$. The algebra $Z_\alpha := Zz_\alpha = A^L_\alpha \cap A^R_\alpha$ is an Abelian division algebra over the ground field k, that is Z_α is a subfield in the center of the separable algebra A^T_α, hence Z_α is a finite separable field extension of k [14, p. 191].

Now we prove that $\widehat{A^T_\alpha}$, the dual of the WHA A^T_α, is isomorphic to the simple k-algebra $M_{n_\alpha}(Z_\alpha)$, where $n_\alpha = \dim_{Z_\alpha} A^R_\alpha$, i.e., A^T_α is simple as a k-coalgebra. We stress that the inclusion $(\widehat{A^T_\alpha})^T \subset \widehat{A^T_\alpha}$ is proper in general. Therefore, simplicity of $\widehat{A^T_\alpha}$ as an algebra is a 'non-trivial' property in the sense that it goes for a WHA which is not trivial, i.e., not generated by the canonical coideal subalgebras $(\widehat{A^T_\alpha})^L$ and $(\widehat{A^T_\alpha})^R$.

Consider the cyclic left $\widehat{A^T_\alpha}$-module $(A^T_\alpha)^R = A^R_\alpha$ with the Sweedler action $\varphi \cdot x^R := \varphi \rightharpoonup x^R$; $\varphi \in \widehat{A^T_\alpha}$, $x^R \in A^R_\alpha$. It is just the trivial representation [2, p. 401] of the WHA $\widehat{A^T_\alpha}$; hence, its endomorphism ring $\mathrm{End}_{\widehat{A^T_\alpha}} A^R_\alpha$ is given by $\widehat{Z}^R_\alpha \rightharpoonup$ [2, p. 402], where $\widehat{Z}^R_\alpha :=$ Center $\widehat{A^T_\alpha} \cap (\widehat{A^T_\alpha})^R$. However, the maps in $\widehat{Z}^R_\alpha \rightharpoonup$ are just multiplications by elements of Z_α due to the statements after (1.6), i.e., $\mathrm{End}_{\widehat{A^T_\alpha}} A^R_\alpha = Z_\alpha$. In addition, A^R_α is a faithful module in this case, because $\varphi \rightharpoonup A^R_\alpha = 0$ implies $\varphi = 0$ due to

$$\begin{aligned} 0 &= \varepsilon\big((\varphi \rightharpoonup A^R_\alpha)A^L_\alpha\big) = \varepsilon\big(\mathbf{1}^{(1)}_\alpha A^L_\alpha\big)\big\langle\varphi, A^R_\alpha \mathbf{1}^{(2)}_\alpha\big\rangle = \big\langle\varphi, A^R_\alpha \Pi^L\big(A^L_\alpha\big)\big\rangle = \big\langle\varphi, A^R_\alpha A^L_\alpha\big\rangle \\ &= \big\langle\varphi, A^T_\alpha\big\rangle, \end{aligned} \tag{2.18}$$

where we used (1.4) and (1.3). Hence, $\widehat{A^T_\alpha}$ is isomorphic to the unital subalgebra $\phi(\widehat{A^T_\alpha}) := \widehat{A^T_\alpha} \rightharpoonup$ of $\mathrm{End}_k A^R_\alpha$. Therefore, the relations $\widehat{A^T_\alpha} \simeq \phi(\widehat{A^T_\alpha}) \subset \mathrm{BiEnd}_{\widehat{A^T_\alpha}} A^R_\alpha = \mathrm{End}_{Z_\alpha} A^R_\alpha \simeq M_{n_\alpha}(Z_\alpha)$ together with the equality

$$\begin{aligned} \dim_k \widehat{A^T_\alpha} &= \dim_k A^T_\alpha = \dim_k\big(A^L_\alpha \otimes_{Z_\alpha} A^R_\alpha\big) = \big(\dim_{Z_\alpha} A^R_\alpha\big)^2 \dim_k Z_\alpha \\ &= n^2_\alpha \dim_k Z_\alpha = \dim_k M_{n_\alpha}(Z_\alpha) \end{aligned} \tag{2.19}$$

concerning the k-dimensions imply the k-algebra isomorphism $\widehat{A^T_\alpha} \simeq M_{n_\alpha}(Z_\alpha)$. $\square$

Theorem 2.4. *The unit left A-module ${}_A A^L$ is the direct sum of simple submodules*

$${}_A A^L = \bigoplus_p {}_A A^L_p, \quad A^L_p := A^L z^L_p, \tag{2.20}$$

where $\{z^L_p\}_p$ is the set of primitive orthogonal idempotents in $Z^L := A^L \cap \mathrm{Center}\, A$.

Proof. Let N be the radical of A. Since N is an ideal in A, we have $N \cdot A^L := \Pi^L(NA^L) \subset \Pi^L(N)$. Due to the identity $N = (\widehat{C}_0)^\perp := \{a \in A \mid \langle\widehat{C}_0, a\rangle = 0\}$ [17, p. 183], where $\widehat{C}_0$ is the coradical of the dual weak Hopf algebra $\widehat{A}$, the previous lemma leads to the containment $N = (\widehat{C}_0)^\perp \subset (\widehat{A}^T)^\perp \subset (\widehat{A}^L)^\perp$. Hence, using (1.7) the canonical pairing gives rise to

$$\big\langle\widehat{A}, \Pi^L(N)\big\rangle = \big\langle\widehat{\Pi}^L(\widehat{A}), N\big\rangle = \big\langle\widehat{A}^L, N\big\rangle = 0, \tag{2.21}$$

i.e., $\Pi^L(N) = 0$. Therefore, the radical of A is in the annihilator ideal of the left module ${}_AA^L$, that is ${}_AA^L$ is semisimple [6, p. 118].

The endomorphism ring for the unit module is given by $\mathrm{End}\,{}_AA^L = Z^L\cdot$ [2, p. 402], that is by the restriction of the A-action to the subalgebra Z^L. Since the unit module is a free, hence faithful A^L-module, it is also faithful as a Z^L-module. Now, the direct sum decomposition (2.20) is clear and $\mathrm{End}\,{}_AA^L_p = Z^L z^L_p\,\cdot$. But $Z^L z^L_p$ is a field, therefore ${}_AA^L_p$ is indecomposable [6, p. 121]. Together with semisimplicity, this leads to simplicity of the direct summands ${}_AA^L_p$. $\square$

The analogous result holds for the unit right A-module.

Remark 2.5. The unit right A-module A^R_A given in Definition 2.1 is the direct sum of simple submodules

$$A^R_A = \bigoplus_p A^R_{pA}, \quad A^R_p := A^R z^R_p, \tag{2.22}$$

where $\{z^R_p := S(z^L_p)\}_p$ is the set of primitive orthogonal idempotents in $Z^R = S(Z^L)$.

We have seen that the simple submodules of the unit left (right) A-module are labelled by primitive idempotents in Z^L (Z^R). Although a generic A-module does not need to be semisimple, it is always a direct sum of submodules labelled by pairs of primitive orthogonal idempotents in the cartesian product $Z^L \times Z^R$. Indeed, the product of primitive orthogonal idempotents in Z^L and Z^R gives rise to a decomposition of the unit

$$\mathbf{1} = \sum_{p,q} z^L_p z^R_q \equiv \sum_{p,q} z^L_p S\big(z^L_q\big)$$

in $Z^L \vee Z^R \subset A^T \cap \mathrm{Center}\, A$.[2] Since $Z^L \vee Z^R \simeq Z^L \otimes_H Z^R$ certain products $z^L_p z^R_q$ can be identically zero due to the presence of the hypercenter $H := Z^L \cap Z^R = A^L \cap A^R \cap \mathrm{Center}\, A$ in A^T. If $z^L_p z^R_q \neq 0$ we refer to (p, q) as an *admissible pair*. Hence, the non-zero summands are labelled by admissible pairs in the decomposition of the unit, which induces a direct sum decomposition of every A-module ${}_AM$ into submodules as $M = \bigoplus_{(p,q)} M_{(p,q)}$, where $M_{(p,q)} := z^L_p z^R_q \cdot M$. We will call the left A-module ${}_AM$ a *member of the* (p, q) *class* and write ${}_AM_{(p,q)}$ if (p, q) is an admissible pair and

$$z^L_{p'} z^R_{q'} \cdot M = \delta_{pp'}\delta_{qq'} M, \tag{2.23}$$

for all product idempotents $z^L_{p'} z^R_{q'} \in Z^L \vee Z^R$. Clearly, the simple submodule ${}_AA^L_p$ of the unit module ${}_AA^L$ is in the 'diagonal' class (p, p), because

$$z^L_q z^R_r \cdot x z^L_p := z^L_q x z^L_p S\big(z^R_r\big) = z^L_q x z^L_p z^L_r = \delta_{p,q}\delta_{p,r} x z^L_p, \quad x \in A^L. \tag{2.24}$$

[2] Note that $S^2\big(z^{L/R}_p\big) = z^{L/R}_p$, because S^2 is inner on $A^{L/R}$ and the idempotents are central.

The next lemma shows, that the simple submodules of the unit module ${}_A A^L$ obey a kind of minimality property in the corresponding class of left A-modules.

Lemma 2.6.

(i) *The k-dimension $|M_{(p,q)}|$ of a nonzero left A-module ${}_A M_{(p,q)}$ in the (p,q) class obeys the inequality*

$$|M_{(p,q)}| \geqslant \max\{|A_p^L|, |A_q^L|\}. \tag{2.25}$$

(ii) *The restriction of A to the subalgebras A_p^L and A_q^R makes ${}_A M_{(p,q)}$ a faithful left A_p^L- and A_q^R-modules, respectively.*

Proof. In the following, first we prove that the left A-modules $M_{(p_1,q_1)}$ and $N_{(p_2,q_2)}$ should obey the matching condition $q_1 = p_2$ in order to get a nonzero product module $M_{(p_1,q_1)} \times N_{(p_2,q_2)}$. Then writing a left A-module $M_{(p,q)}$ as a product with the unit module and using this matching condition, the emerging tensor product space can be given as a sum of subspaces with respect to a basis of the corresponding simple submodule of the unit module. We will use Theorem 2.4 and Remark 2.5 to prove that $M_{(p,q)}$ is a faithful A_p^L- and A_q^R-module and then the estimation of the k-dimension of $M_{(p,q)}$ will follow.

Using property (1.12) of the separating idempotent of A^L and the decomposition of the unit into primitive orthogonal idempotents in Z^L, one obtains

$$\Delta(\mathbf{1}) = \sum_r S^{-1}\big(S(\mathbf{1}^{(1)})\big) \otimes \mathbf{1}^{(2)} z_r^L = \sum_r \big(z_r^R \otimes z_r^L\big)\Delta(\mathbf{1}). \tag{2.26}$$

Therefore, for any two left A-modules M, N within a certain class, we have

$$\begin{aligned} z_p^L z_q^R \cdot M_{(p_1,q_1)} \times N_{(p_2,q_2)} &= z_p^L z_q^R \cdot \big(\Delta(\mathbf{1})\big(M_{(p_1,q_1)} \otimes N_{(p_2,q_2)}\big)\big) \\ &= \sum_r z_p^L z_r^R \mathbf{1}^{(1)} \cdot M_{(p_1,q_1)} \otimes z_r^L z_q^R \mathbf{1}^{(2)} \cdot N_{(p_2,q_2)}, \end{aligned} \tag{2.27}$$

implying

$${}_A M_{(p_1,q_1)} \times {}_A N_{(p_2,q_2)} = \begin{cases} 0, & q_1 \neq p_2, \\ {}_A(M \times N)_{(p_1,q_2)}, & q_1 = p_2. \end{cases} \tag{2.28}$$

The separating idempotent of A^L is a quasibasis for the counit due to (1.13), hence, it has the expression $S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)} = \sum_i f_i \otimes e_i$, where $\{e_i\}_i, \{f_i\}_i \subset A^L$ are dual bases with respect to the counit $\varepsilon(e_i f_j) = \delta_{i,j}$. Choosing a basis $\{e_i\}_i = \bigcup_p \{e_i\}_{i \in p}$ that respects the direct sum decomposition $A^L = \bigoplus_p A_p^L$, i.e., $\{e_i\}_{i\in p} \subset A_p^L$, $f_i \otimes e_i \in A_p^L \otimes A_p^L$, $i \in p$ follows. Since X_M^L and X_M^R defined in (2.3) are left A-module isomorphisms,

$$\begin{aligned}\left|M_{(p,q)}\right| &= \left|A^L \times M_{(p,q)}\right| = \left|S\big(\mathbf{1}^{(1)}\big) \otimes \mathbf{1}^{(2)} z_p^L \cdot M_{(p,q)}\right| \\ &= \left|\sum_{i\in p} f_i \otimes e_i \cdot M_{(p,q)}\right| = \sum_{i\in p} \left|e_i \cdot M_{(p,q)}\right|, \qquad (2.29\text{a})\\ \left|M_{(p,q)}\right| &= \left|M_{(p,q)} \times A^L\right| = \left|\mathbf{1}^{(1)} z_q^R \cdot M_{(p,q)} \otimes \mathbf{1}^{(2)}\right| \\ &= \left|\sum_{i\in q} S^{-1}(f_i) \cdot M_{(p,q)} \otimes e_i\right| = \sum_{i\in q} \left|S^{-1}(f_i) \cdot M_{(p,q)}\right| \qquad (2.29\text{b})\end{aligned}$$

for any left A-module $M_{(p,q)}$ in the (p,q) class. Hence, if we prove that $M_{(p,q)}$ is a faithful left A_p^L- and A_q^R-module, i.e., $x_p^L \cdot M_{(p,q)}$ and $x_q^R \cdot M_{(p,q)}$ are nonzero linear subspaces of $M_{(p,q)}$ for all non-zero elements $x_p^L \in A_p^L$ and $x_q^R \in A_q^R$, respectively, then we are done, because a nonzero linear subspace is at least one-dimensional and $|A_q^R| = |S(A_q^R)| = |A_q^L|$ due to the invertibility of the antipode S.

Let us suppose, that $0 \neq x_p^L \in A_p^L$ is in the annihilator ideal of ${}_AM_{(p,q)}$. Since in the simple module ${}_AA_p^L$ every non-zero element is cyclic,

$$A_p^L = \left\{a \cdot x_p^L := a^{(1)} x_p^L S\big(a^{(2)}\big) \;\middle|\; a \in A\right\}$$

should also be contained in the annihilator ideal of ${}_AM_{(p,q)}$. But this contradicts the assumption that ${}_AM_{(p,q)}$ is a nonzero module in the (p,q) class. Since the module A^R_{qA} is simple (see Remark 2.5), one has

$$A^R_{qA} = \left\{x_q^R \cdot a := S\big(a^{(1)}\big) x_q^R a^{(2)} \;\middle|\; a \in A\right\}$$

for any non-zero $x_q^R \in A_q^R$. Hence, the assumption that a non-zero element of A_q^R is in the annihilator ideal of ${}_AM_{(p,q)}$ leads to the contradiction as before. □

Corollary 2.7. *Given ${}_AM$ let ${}_{A^L}M$ and ${}_{A^R}M$ denote the A^L- and A^R-module, respectively, defined by restriction of the A-module structure to these subalgebras. If* $\mathrm{End}_{A^R} M = A^L\cdot \simeq A^L$*, then ${}_{A^L}M$ and ${}_{A^R}M$ are free rank one A^L- and A^R-modules, respectively.*

Proof. Repeating the argument in [2, p. 417], one obtains an upper bound for the k-dimension $|M|$ of the module M: being separable, A^R is semisimple; hence, by the Wedderburn structure theorem $A^R = \bigoplus_i A_i^R \simeq \bigoplus_i M_{n_i}(D_i)$, where D_i is a division algebra corresponding to the simple ideal A_i^R. Let m_i denote the multiplicity of the simple A_i^R-submodules in the semisimple module ${}_{A^R}M$. Then $\mathrm{End}_{A^R} M \simeq \bigoplus_i M_{m_i}(D_i)$, which is isomorphic to A^L by assumption. Hence, as a right action on M, it is antiisomorphic to A^L, i.e., isomorphic to A^R. This is possible only if there is a permutation σ of simple ideals of A^R such that $n_{\sigma(i)} = m_i$ and $D_{\sigma(i)} = D_i$. Therefore, $M = \bigoplus_i \mathrm{Mat}(n_i \times \sigma(n_i), D_i)$ as an A^R-bimodule and $|M| = \sum_i |D_i| n_i n_{\sigma(i)}$. The upper bound $|A^R| = \sum_i |D_i| n_i^2$ for $|M|$ follows from the Cauchy–Schwarz inequality.

However, the A^R-bimodule structure of M implies that M is a faithful left A^R-module, hence, a faithful left Z^R-module. Therefore, the previous lemma leads to the opposite estimation: $|M| \geqslant |A^R|$. Thus

$$\sum_i |D_i| n_i n_{\sigma(i)} = |M| = \left|A^R\right| = \sum_i |D_i| n_i^2,$$

which is possible only if $n_{\sigma(i)} = n_i$. But in this case ${}_{A^R}M$ and ${}_{A^L}M$ are isomorphic to the left regular A^R- and A^L-module, respectively, that is ${}_AM$ becomes a free rank one A^R- and A^L-module by restricting the A-action to these subalgebras. □

Lemma 2.8. *If the isomorphism $A^L \simeq M \times \overleftarrow{M}$ ($A^L \simeq \overrightarrow{M} \times M$) of left A-modules holds for a left A-module ${}_AM$, where A^L denotes the unit left A-module, then* $\operatorname{End}_{A^R} M = A^L \cdot \simeq A^L$ ($\operatorname{End}_{A^L} M = A^R \cdot \simeq A^R$) *as k-algebras.*

Proof. Since $\overleftarrow{M} := \operatorname{Hom}_k(M, k)$ as a k-linear space, one can realize $\operatorname{End}_k M$ by $M \otimes \overleftarrow{M}$ as

$$\left(\sum_a m_a \otimes \widehat{m}_a\right)(m) := \sum_a m_a \langle \widehat{m}_a, m\rangle, \quad m \in M.$$

The subalgebra $\operatorname{End}_{A^R} M \subset \operatorname{End}_k M$ is given by $M \times \overleftarrow{M}$: if $f = \sum_a m_a \otimes \widehat{m}_a \in \operatorname{End}_{A^R} M$ then using (2.6) we get

$$\begin{aligned} f(m) &= f(\mathbf{1}\cdot m) = f\big(\mathbf{1}^{(1)} S\big(\mathbf{1}^{(2)}\big)\cdot m\big) = \mathbf{1}^{(1)} \cdot f\big(S\big(\mathbf{1}^{(2)}\big)\cdot m\big) \\ &= \sum_a \mathbf{1}^{(1)} \cdot m_a \big\langle \widehat{m}_a, S\big(\mathbf{1}^{(2)}\big)\cdot m\big\rangle = \sum_a \mathbf{1}^{(1)}\cdot m_a \big\langle \mathbf{1}^{(2)}\cdot \widehat{m}_a, m\big\rangle \\ &= \left(\sum_a \mathbf{1}^{(1)} \cdot m_a \otimes \mathbf{1}^{(2)} \cdot \widehat{m}_a\right)(m), \quad m \in M, \end{aligned} \tag{2.30}$$

that is $\operatorname{End}_{A^R} M \subset M \times \overleftarrow{M}$. Choosing $f = \sum_a m_a \otimes \widehat{m}_a \in M \times \overleftarrow{M}$ and $x^R \in A^R$,

$$\begin{aligned} f\big(x^R \cdot m\big) &\equiv \left(\sum_a \mathbf{1}^{(1)}\cdot m_a \otimes \mathbf{1}^{(2)} \cdot \widehat{m}_a\right)\big(x^R\cdot m\big) = \sum_a \mathbf{1}^{(1)}\cdot m_a \big\langle \mathbf{1}^{(2)}\cdot \widehat{m}_a, x^R \cdot m\big\rangle \\ &= \sum_a \mathbf{1}^{(1)} \cdot m_a \big\langle S^{-1}\big(x^R\big)\mathbf{1}^{(2)}\cdot \widehat{m}_a, m\big\rangle = \sum_a x^R \mathbf{1}^{(1)}\cdot m_a \big\langle \mathbf{1}^{(2)}\cdot\widehat{m}_a, m\big\rangle \\ &= x^R \cdot \left(\sum_a m_a \otimes \widehat{m}_a\right)(m) = x^R \cdot f(m), \quad m \in M, \end{aligned} \tag{2.31}$$

leads to the opposite containment, hence $\operatorname{End}_{A^R} M = M \times \overleftarrow{M}$.

The structure of the subalgebra $M \times \overleftarrow{M}$ of $M \otimes \overleftarrow{M}$ can be obtained as follows. The unit of $\operatorname{End}_k M$ is given by $1_M = \sum_i m_i \otimes \widehat{m}_i \in M \otimes \overleftarrow{M}$, where $\{m_i\}_i \subset M$ and $\{\widehat{m}_i\}_i \subset \overleftarrow{M}$

are dual bases. 1_M is in $M \times \overleftarrow{M}$ due to (2.7b). Let $U : A^L \to M \times \overleftarrow{M}$ be the required left A-module isomorphism. Hence, there exists $u^L \in A^L$ such that $1_M = U(u^L)$ and

$$\begin{aligned} \operatorname{End}_{A^R} M &= M \times \overleftarrow{M} = U\big(A^L\big) = U\big(A^L \cdot \mathbf{1}\big) = A^L \cdot U(\mathbf{1}) \\ &=: A^L \cdot \left(\sum_a m_a \otimes \widehat{m}_a\right) = \sum_i A^L \cdot m_a \otimes \widehat{m}_a = \big(A^L \cdot\big) \circ U(\mathbf{1}), \end{aligned} \tag{2.32}$$

using (1.4) in the sixth equality. Since the unit A-module ${}_A A^L$ becomes a free, hence faithful left A^L-module by restriction and since $U(\mathbf{1}) \in \operatorname{End}_{A^R} M$ is invertible due to $1_M = U(u^L) = (u^L \cdot) \circ U(\mathbf{1})$, (2.32) leads to $\operatorname{End}_{A^R} M = A^L \cdot \simeq A^L$.

The proof of the statement involving the right dual $\overrightarrow{M}$ is similar. $\square$

3. Hopf modules in weak Hopf algebras

Besides A-modules, we need the notion of weak Hopf modules of a WBA A [2, p. 407]. First, a *left* (*right*) *A-comodule* is a pair ${}^A M \equiv (M, \delta_L)$ ($M^A \equiv (M, \delta_R)$) consisting of a finite dimensional k-linear space and a k-linear map $\delta_L : M \to A \otimes M$ ($\delta_R : M \to M \otimes A$) called the coaction that obeys

$$\begin{aligned} &(\mathrm{id}_A \otimes \delta_L) \circ \delta_L = (\Delta \otimes \mathrm{id}_M) \circ \delta_L, \qquad && (\delta_R \otimes \mathrm{id}_A) \circ \delta_R = (\mathrm{id}_M \otimes \Delta) \circ \delta_R, \\ &(\varepsilon \otimes \mathrm{id}_M) \circ \delta_L = \mathrm{id}_M, && (\mathrm{id}_M \otimes \varepsilon) \circ \delta_R = \mathrm{id}_M. \end{aligned} \tag{3.1}$$

They incorporate only the coalgebra properties of A. In the following, we will use the notations $\delta_L(m) \equiv m_{-1} \otimes m_0$ and $\delta_R(m) \equiv m_0 \otimes m_1$. Lower and upper A-indices will indicate A-modules and A-comodules, respectively.

The *weak Hopf modules* (WHM) M_A^A, ${}_A M^A$, ${}_A^A M$, ${}^A M_A$ of a WBA A are A-modules and A-comodules simultaneously together with a compatibility condition restricting the comodule map to be an A-module map, e.g.,

$$\begin{aligned} &M_A^A \equiv (M, \mu_R, \delta_R): \quad (m \cdot a)_0 \otimes (m \cdot a)_1 = m_0 \cdot a^{(1)} \otimes m_1 a^{(2)}, \\ &{}_A M^A \equiv (M, \mu_L, \delta_R): \quad (a \cdot m)_0 \otimes (a \cdot m)_1 = a^{(1)} \cdot m_0 \otimes a^{(2)} m_1, \quad a \in A,\ m \in M. \end{aligned} \tag{3.2}$$

As a consequence of these identities, WHMs obey a kind of non-degeneracy property

$$\begin{aligned} &m = m_0 \cdot \Pi^R(m_1), \quad m \in M_A^A; \qquad && m = \overline{\Pi}^R(m_1) \cdot m_0, \quad m \in {}_A M^A; \\ &m = m_0 \cdot \overline{\Pi}^L(m_{-1}), \quad m \in {}^A M_A; && m = \Pi^L(m_{-1}) \cdot m_0, \quad m \in {}_A^A M. \end{aligned} \tag{3.3}$$

We call ${}_A M_A^A$, ${}^A M_A^A$, ${}_A^A M_A$, ${}_A^A M^A$, ${}_A^A M_A^A$ *multiple weak Hopf modules* if they are pairwise WHMs of the WBA A in the possible A-indices and if the different module or comodule maps commute, i.e., they are bimodules or bicomodules. The *invariants* and *coinvariants* of left/right A-modules and left/right A-comodules, respectively, are defined to be

$$\begin{aligned}
I({}_A M) &:= \{m \in M \mid a \cdot m = \pi^L(a) \cdot m,\ a \in A\},\\
I(M_A) &:= \{m \in M \mid m \cdot a = m \cdot \pi^R(a),\ a \in A\},\\
C({}^A M) &:= \{m \in M \mid \delta_L(m) \in A^R \otimes M\},\\
C(M^A) &:= \{m \in M \mid \delta_R(m) \in M \otimes A^L\}.
\end{aligned} \tag{3.4}$$

For example, the left/right invariants and the left/right coinvariants of the multiple weak Hopf module ${}^A_A A^A_A \equiv (A, \mu, \mu, \Delta, \Delta)$ of a WBA A are the left/right integrals $I^{L/R}$ and the right/left subalgebras $A^{R/L}$, respectively. Dualizing left/right actions or coactions of a WBA A with the help of dual bases in A and $\widehat{A}$ with respect to the canonical pairing, one arrives at right/left coactions or actions of the dual WBA $\widehat{A}$, respectively, e.g.,

$${}^{\widehat{A}}M\colon \quad \hat{\delta}_L(m) := \beta_i \otimes m \cdot b_i, \quad m \in M_A, \tag{3.5a}$$

$${}_{\widehat{A}}M\colon \quad \varphi \cdot m := m_0\langle \varphi, m_1\rangle, \quad m \in M^A,\ \varphi \in \widehat{A}, \tag{3.5b}$$

and the invariants (coinvariants) with respect to A become coinvariants (invariants) with respect to $\widehat{A}$.

If A is not only a WBA, but also a WHA, one can say more about the invariants and coinvariants of (multiple) WHMs.

Lemma 3.1. *Let A be a WHA.*

(i) *The coinvariants and the invariants of a WHM of A can be equivalently characterized as follows*:

$$\begin{aligned}
C(M^A_A) &= \{m \in M \mid \delta_R(m) = m \cdot \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)}\},\\
C({}_A M^A) &= \{m \in M \mid \delta_R(m) = \mathbf{1}^{(1)} \cdot m \otimes \mathbf{1}^{(2)}\},\\
C({}^A_A M) &= \{m \in M \mid \delta_L(m) = \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)} \cdot m\},\\
C({}^A M_A) &= \{m \in M \mid \delta_L(m) = \mathbf{1}^{(1)} \otimes m \cdot \mathbf{1}^{(2)}\},
\end{aligned} \tag{3.6a}$$

$$\begin{aligned}
I(M^A_A) &= \{m \in M \mid m_0 \cdot a \otimes m_1 = m_0 \otimes m_1 S(a),\ a \in A\},\\
I({}_A M^A) &= \{m \in M \mid m_0 \otimes a m_1 = S(a) \cdot m_0 \otimes m_1,\ a \in A\},\\
I({}^A_A M) &= \{m \in M \mid m_{-1} \otimes a \cdot m_0 = S(a) m_{-1} \otimes m_1,\ a \in A\},\\
I({}^A M_A) &= \{m \in M \mid m_{-1} a \otimes m_0 = m_{-1} \otimes m_0 \cdot S(a),\ a \in A\}.
\end{aligned} \tag{3.6b}$$

(ii) *The following maps define projections from WHMs onto their coinvariants and invariants, respectively*:

$$P^A(m) := m_0 \cdot S(m_1), \quad m \in M_A^A, \qquad \overline{P}^A(m) := S^{-1}(m_1) \cdot m_0, \quad m \in {}_AM^A,$$
$${}^AP(m) := S(m_{-1}) \cdot m_0, \quad m \in {}^A_AM, \qquad {}^A\overline{P}(m) := m_0 \cdot S^{-1}(m_{-1}), \quad m \in {}^AM_A, \tag{3.7a}$$

$$P_A(m) := m_0 \cdot R(m_1), \quad m \in M_A^A, \qquad {}_A\overline{P}(m) := \overline{L}(m_1) \cdot m_0, \quad m \in {}_AM^A,$$
$${}_AP(m) := L(m_{-1}) \cdot m_0, \quad m \in {}^A_AM, \qquad \overline{P}_A(m) := m_0 \cdot \overline{R}(m_{-1}), \quad m \in {}^AM_A, \tag{3.7b}$$

where S is the antipode and R, $\overline{R}$, L, $\overline{L}$ are the projection maps (1.17) *to integrals in the WHA A.*

(iii) *In case of the multiple WHMs ${}_AM_A^A$ and ${}^A_AM_A$ the coinvariants are left and right A-modules with respect to the induced left and right adjoint actions, respectively.*

Proof. (i) The characterization (3.6a) of coinvariants and the form (3.7a) of the projections onto them have been already proved in [2, p. 409]. Concerning the invariants of M_A^A, first we note that the set given in (3.6b) is contained in the set of invariants defined in (3.4) since

$$\begin{aligned} m \cdot a &= (\mathrm{id} \otimes \varepsilon)\big(m_0 \cdot a^{(1)} \otimes m_1 a^{(2)}\big) = (\mathrm{id} \otimes \varepsilon)\big(m_0 \otimes m_1 S\big(a^{(1)}\big)a^{(2)}\big) \\ &= (\mathrm{id} \otimes \varepsilon)\big(m_0 \otimes m_1 \Pi^R(a)\big) = (\mathrm{id} \otimes \varepsilon)\big(\delta_R\big(m \cdot \Pi^R(a)\big)\big) = m \cdot \Pi^R(a), \end{aligned} \tag{3.8}$$

for all $a \in A$. Using the third identity in (1.8), the opposite containment is as follows

$$\begin{aligned} m_0 \cdot a \otimes m_1 &= m_0 \cdot \mathbf{1}^{(1)} a \otimes m_1 \mathbf{1}^{(2)} = m_0 \cdot a^{(1)} \otimes m_1 \Pi^L\big(a^{(2)}\big) \\ &= m_0 \cdot a^{(1)} \otimes m_1 a^{(2)} S\big(a^{(3)}\big) = \big(m \cdot a^{(1)}\big)_0 \otimes \big(m \cdot a^{(1)}\big)_1 S\big(a^{(2)}\big) \\ &= \big(m \cdot \Pi^R\big(a^{(1)}\big)\big)_0 \otimes \big(m \cdot \Pi^R\big(a^{(1)}\big)\big)_1 S\big(a^{(2)}\big) = m_0 \otimes m_1 \Pi^R\big(a^{(1)}\big) S\big(a^{(2)}\big) \\ &= m_0 \otimes m_1 S(a), \quad a \in A, \ m \in I\big(M_A^A\big). \end{aligned} \tag{3.9}$$

The cases of the other three WHMs can be proved similarly.

(ii) The image of the map P_A is in $I(M_A^A)$ due to the defining property (1.9) of the right integrals in A. Applying P_A to an invariant $m \in I(M_A^A)$ and using their characterization (3.6b) and the non-degeneracy property (3.3),

$$\begin{aligned} m_0 \cdot R(m_1) &:= m_0 \cdot \big((m_1 b_i) \leftharpoonup \widehat{S}^2(\beta_i)\big) = m_0 \cdot \big(\big(m_1 S(b_i)\big) \leftharpoonup \widehat{S}(\beta_i)\big) \\ &= (m_0 \cdot b_i) \cdot \big(m_1 \leftharpoonup \widehat{S}(\beta_i)\big) = m_0 \cdot b_i m_1^{(2)} \big\langle m_1^{(1)}, \widehat{S}(\beta_i)\big\rangle \\ &= m_0 \cdot S\big(m_1^{(1)}\big) m_1^{(2)} = m_0 \cdot \Pi^R(m_1) = m, \quad m \in I\big(M_A^A\big) \end{aligned} \tag{3.10}$$

follows, that is P_A is a projection onto the invariants of M_A^A. The cases of projections onto the invariants of the other three WHMs can be proved similarly.

(iii) We have to show that the maps

$$\nu_L(a\otimes m)\equiv a\star m := a^{(1)}\cdot m\cdot S\big(a^{(2)}\big),\quad a\in A,\ m\in C\big({}_AM_A^A\big),$$
$$\nu_R(m\otimes a)\equiv m\star a := S\big(a^{(1)}\big)\cdot m\cdot a^{(2)},\quad a\in A,\ m\in C\big({}_A^AM_A\big) \tag{3.11}$$

provide a left and a right A-module structure $(C({}_AM_A^A),\nu_L)$ and $(C({}_A^AM_A),\nu_R)$, respectively. The image of the map ν_L is in $C({}_AM_A^A)$, because for all $a\in A$ and $m\in C({}_AM_A^A)$

$$\begin{aligned}\delta_R\big(a^{(1)}\cdot m\cdot S\big(a^{(2)}\big)\big) &= a^{(11)}\cdot\big(\mathbf{1}^{(1)}\cdot m\big)\cdot S\big(a^{(2)}\big)^{(1)}\otimes a^{(12)}\mathbf{1}^{(2)}S\big(a^{(2)}\big)^{(2)}\\ &= a^{(1)}\cdot m\cdot S\big(a^{(4)}\big)\otimes a^{(2)}S\big(a^{(3)}\big)\\ &= a^{(1)}\cdot m\cdot S\big(a^{(3)}\big)\otimes\Pi^L\big(a^{(2)}\big)\in M\otimes A^L.\end{aligned} \tag{3.12}$$

The map ν_L is clearly a left A-action, i.e., $a\star(b\star m)=ab\star m$, for all $a,b\in A$ and $m\in C({}_AM_A^A)$, moreover, for all $m\in C({}_AM_A^A)$

$$\begin{aligned}m=\mathbf{1}\cdot m &= (\mathbf{1}\cdot m)_0\cdot\Pi^R\big((\mathbf{1}\cdot m)_1\big)=\mathbf{1}^{(1)}\cdot m_0\cdot\Pi^R\big(\mathbf{1}^{(2)}m_1\big)\\ &= \mathbf{1}^{(1)}\cdot m\cdot\Pi^R\big(\mathbf{1}^{(2)}\big)=\mathbf{1}^{(1)}\cdot m\cdot S\big(\mathbf{1}^{(2)}\big)=\mathbf{1}\star m,\end{aligned} \tag{3.13}$$

where we used the identities (3.3) and (3.2), the property (3.6a) of the coinvariants, and (1.10). The proof of the case $(C({}_A^AM_A),\nu_R)$ is similar. □

Extending the result of [2, p. 410] concerning the structure of a WHM, the structure of a multiple WHM is given by the following.

Theorem 3.2.

(i) *Let ${}_AM_A^A$ be a multiple weak Hopf module of the WHA A. Then ${}_AM_A^A$ is isomorphic as a multiple WHM to ${}_A(C(M)\times A_A^A)$, which as a left A-module is isomorphic to the product of the left A-modules $(C(M),\star)$ defined in the previous lemma and the left regular module ${}_AA$, while the right A-module and right A-comodule structures are inherited from the WHM $A_A^A\equiv(A,\mu,\Delta)$.*

(ii) *In particular, ${}_A\widehat{A}_A^A\equiv(\widehat{A},\mu_L,\mu_R,\delta_R)$ is a multiple WHM with structure maps*

$$\mu_L(a\otimes\varphi)\equiv a\cdot\varphi := \varphi\leftharpoonup S^{-1}(a), \tag{3.14a}$$
$$\mu_R(\varphi\otimes a)\equiv\varphi\cdot a := S(a)\rightharpoonup\varphi,\quad a\in A,\ \varphi\in\widehat{A}, \tag{3.14b}$$
$$\delta_R(\varphi)\equiv\varphi_0\otimes\varphi_1 := \beta_i\varphi\otimes b_i, \tag{3.14c}$$

where $\{b_i\}\subset A$ and $\{\beta_i\}\subset\widehat{A}$ are dual bases with respect to the canonical pairing, therefore,

$$ {}_A\widehat{A}_A^A \simeq {}_A\big(C\big(\widehat{A}\big) \times A_A^A\big) = {}_A\big(\hat{I}^L \times A_A^A\big), \tag{3.15}$$

where $\hat{I}^L$ is the space of left integrals in the WHA $\widehat{A}$.

Proof. (i) As a k-linear space ${}_A(C(M) \times A_A^A) \equiv (C(M) \times A, \mu_L, \mu_R, \delta_R)$ is (see (2.2a))

$$\begin{aligned} C(M) \times A &:= \mathbf{1}^{(1)} \star C(M) \otimes \mathbf{1}^{(2)} A = \mathbf{1}^{(1)} \cdot C(M) \cdot S\big(\mathbf{1}^{(2)}\big) \otimes \mathbf{1}^{(3)} A \\ &= C(M) \cdot \mathbf{1}^{(1)} S\big(\mathbf{1}^{(2)}\big) \otimes \mathbf{1}^{(3)} A = C(M) \cdot \Pi^L\big(\mathbf{1}^{(1)}\big) \otimes \mathbf{1}^{(2)} A \\ &= C(M) \cdot S\big(\mathbf{1}^{(1)}\big) \otimes \mathbf{1}^{(2)} A \end{aligned} \tag{3.16}$$

due to the fact that

$$x^R \cdot m = x^R \cdot (\mathbf{1} \star m) := x^R \mathbf{1}^{(1)} \cdot m \cdot S\big(\mathbf{1}^{(2)}\big) = m \cdot x^R, \quad m \in C(M),\ x^R \in A^R, \tag{3.17}$$

which follows from the identities (3.13) and (1.12). One can easily check, that the maps

$$\begin{aligned} &a \cdot \left(\sum_i n_i \otimes b_i\right) := \sum_i a^{(1)} \star n_i \otimes a^{(2)} b_i, \qquad \left(\sum_i n_i \otimes b_i\right) \cdot a := \sum_i n_i \otimes b_i a, \\ &\delta_R\left(\sum_i n_i \otimes b_i\right) := \sum_i n_i \otimes b_i^{(1)} \otimes b_i^{(2)}, \quad a \in A,\ \sum_i n_i \otimes b_i \in C(M) \times A \end{aligned} \tag{3.18}$$

provide $C(M) \times A$ with a multiple WHM-structure. The k-linear maps $U : {}_A M_A^A \to C(M) \times A$ and $V : C(M) \times A \to {}_A M_A^A$ defined by

$$U(m) := m_0 \cdot S(m_1) \otimes m_2, \qquad V\left(\sum_i n_i \otimes b_i\right) := \sum_i n_i \cdot b_i \tag{3.19}$$

are inverses of each other [2, p. 411], i.e., $V \circ U = \mathrm{id}_M$ and $U \circ V = \mathrm{id}_{C(M)\times A}$. In order to prove that ${}_A M_A^A$ and ${}_A(C(M) \times A_A^A)$ are isomorphic as multiple WHMs as well, we have to show that both U and V are left and right A-module and right A-comodule maps. We can restrict ourselves to the left A-module properties, because the two other properties were already shown in [2, p. 411].

$$\begin{aligned} U(a \cdot m) &= (a \cdot m)_0 \cdot S\big((a \cdot m)_1\big) \otimes (a \cdot m)_2 = a^{(1)} \cdot m_0 \cdot S(m_1) S\big(a^{(2)}\big) \otimes a^{(3)} m_2 \\ &= a^{(1)} \star \big(m_0 \cdot S(m_1)\big) \otimes a^{(2)} m_2 = a \cdot U(m), \quad a \in A,\ m \in M, \end{aligned} \tag{3.20a}$$

$$\begin{aligned} V\left(a \cdot \left(\sum_i n_i \otimes b_i\right)\right) &= \sum_i V\big(a^{(1)} \star n_i \otimes a^{(2)} b_i\big) = \sum_i V\big(a^{(1)} \cdot n_i \cdot S\big(a^{(2)}\big) \otimes a^{(3)} b_i\big) \\ &= \sum_i a^{(1)} \cdot n_i \cdot S\big(a^{(2)}\big) a^{(3)} b_i = \sum_i a^{(1)} \cdot n_i \cdot \Pi^R\big(a^{(2)}\big) b_i \end{aligned}$$

$$= \sum_i a^{(1)} \Pi^R\big(a^{(2)}\big) \cdot n_i \cdot b_i = a \cdot \left(\sum_i n_i \cdot b_i\right)$$
$$= a \cdot V\left(\sum_i n_i \otimes b_i\right), \quad a \in A,\ \sum_i n_i \otimes b_i \in C(M) \times A, \tag{3.20b}$$

where we used (3.17) in the fifth equality of (3.20b).

(ii) The WHM structure $\widehat{A}_A^A \equiv (\widehat{A}, \mu_R, \delta_R)$ given by (3.14b and c) of the multiple WHM ${}_A\widehat{A}_A^A$ has been shown in [2, p. 409]. The map μ_L defined in (3.14a) is clearly a left A-module map on $\widehat{A}$ that commutes with the given right A-module map μ_R. The right comodule map δ_R is also a left A-module map since

$$\begin{aligned}
\delta_R(a \cdot \varphi) &:= \delta_R\big(\varphi \leftharpoonup S^{-1}(a)\big) = \delta_R\big(\varphi \leftharpoonup \overline{\Pi}^L\big(a^{(2)}\big)S^{-1}\big(a^{(1)}\big)\big) \\
&= \delta_R\big(\big(\big(\hat{\mathbf{1}} \leftharpoonup \overline{\Pi}^L\big(a^{(2)}\big)\big)\varphi\big) \leftharpoonup S^{-1}\big(a^{(1)}\big)\big) \\
&= \delta_R\big(\big(\big(\hat{\mathbf{1}} \leftharpoonup a^{(2)}\big)\varphi\big) \leftharpoonup S^{-1}\big(a^{(1)}\big)\big) \\
&= \delta_R\big(\hat{\mathbf{1}} \leftharpoonup a^{(3)}S^{-1}\big(a^{(2)}\big)\big)\big(\varphi \leftharpoonup S^{-1}\big(a^{(1)}\big)\big) \\
&= \beta_i\big(\hat{\mathbf{1}} \leftharpoonup \overline{\Pi}^R\big(a^{(2)}\big)\big)\big(\varphi \leftharpoonup S^{-1}\big(a^{(1)}\big)\big) \otimes b_i \\
&= \big(\beta_i \leftharpoonup \overline{\Pi}^R\big(a^{(2)}\big)\big)\big(\varphi \leftharpoonup S^{-1}\big(a^{(1)}\big)\big) \otimes b_i \\
&= \big(\big(\beta_i \leftharpoonup a^{(2)}\big)\varphi\big) \leftharpoonup S^{-1}\big(a^{(1)}\big) \otimes b_i \\
&= (\beta_i\varphi) \leftharpoonup S^{-1}\big(a^{(1)}\big) \otimes a^{(2)}b_i = a^{(1)} \cdot \varphi_0 \otimes a^{(2)}\varphi_1, \quad a \in A,\ \varphi \in \widehat{A},
\end{aligned} \tag{3.21}$$

where we used the identities (1.6) and (1.10), (1.11). Hence, the maps (3.14) provides $\widehat{A}$ with a multiple WHM structure and the statement (3.15) follows from the previously proved structure of a general multiple weak Hopf module. By dualizing the right A-coaction to left $\widehat{A}$-action as in (3.5b), the right coinvariants $C(\widehat{A}^A)$ become the left invariants of the left regular module ${}_{\widehat{A}}\widehat{A}$, which is the space of left integrals $\hat{I}^L$ in $\widehat{A}$. □

Corollary 3.3. *The left regular A-module ${}_AA$ is injective, i.e., A is a quasi-Frobenius algebra.*

Proof. The inverse of the antipode provides the isomorphism of the right A-modules

$$\widehat{S}^{-1} : \big(\widehat{A}_A, \leftharpoonup\big) \to \big(\widehat{A}_A, \mu_R\big), \tag{3.22}$$

with right action μ_R given in (3.14b) and the structure theorem of multiple weak Hopf modules implies that $(\widehat{A}_A, \mu_R)$ is isomorphic to a direct summand of the free right A-module $\hat{I}^L \otimes A_A$. Therefore, $(\widehat{A}_A, \leftharpoonup)$ is a projective right A-module, which implies the injectivity of its k-dual, that is of ${}_AA$. Hence, A is a quasi-Frobenius algebra [5, p. 414], which has been already established in [2, p. 413]. □

Corollary 3.4. *${}_A I^R$ and A^R_A are A-duals of each other. ${}_A\hat{I}^L$ is the right conjugate module ${}_A\overrightarrow{I^R}$ of ${}_A I^R$ and they are the direct sum of simple submodules*

$$ {}_A I^R = \bigoplus_p {}_A I^R_p, \qquad I^R_p := z^R_p I^R, \tag{3.23a}$$

$$ {}_A \hat{I}^L = \bigoplus_p {}_A \hat{I}^L_p, \qquad \hat{I}^L_p := z^R_p \star \hat{I}^L, \tag{3.23b}$$

where $\{z^R_p\}_p$ is the set of primitive orthogonal idempotents in Z^R.

Proof. Since the right integrals I^R form a left ideal in A and ${}_A A$ is injective by Corollary 3.3, it follows [5, p. 392] that every $\phi \in \mathrm{Hom}({}_A I^R, {}_A A)$ can be extended to $\bar{\phi} \in \mathrm{Hom}({}_A A, {}_A A)$. But every such element $\bar{\phi}$ is given by a right multiplication of an element $a \in A$, hence any ϕ is given by the restriction of a right multiplication to I^R

$$\phi(r) = \bar{\phi}(r) \equiv ra = r\Pi^R(a), \quad r \in I^R.$$

This establish that $\mathrm{Hom}({}_A I^R, {}_A A) \simeq A^R$ as a k-linear space. The isomorphic right A-module structure, $\mathrm{Hom}({}_A I^R, {}_A A)_A \simeq A^R_A$, follows from

$$\begin{aligned}&(\phi_x \cdot a)(r) := \phi_x(r)a = (rx)a = r\Pi^R(xa) =: r(x \cdot a) = \phi_{x\cdot a}(r),\\ &\quad x \in A^R,\ r \in I^R, a \in A.\end{aligned} \tag{3.24}$$

The proof of other duality relation is as follows. A map $f \in \mathrm{Hom}(A^R_A, A_A)$ is just a left multiplication with the image $f(\mathbf{1}) \in A$,

$$f(x) = f\big(\Pi^R(\mathbf{1}x)\big) =: f(\mathbf{1}\cdot x) = f(\mathbf{1})x, \quad x \in A^R, \tag{3.25}$$

which should be a right integral, $f(\mathbf{1}) \in I^R$, because of the module homomorphism property of f and (3.25)

$$f(\mathbf{1})a = f(\mathbf{1}\cdot a) := f\big(\Pi^R(\mathbf{1}a)\big) = f(\mathbf{1})\Pi^R(a). \tag{3.26}$$

The isomorphic left A-module structure of I^R and $\mathrm{Hom}\,(A^R_A, A_A)$ is immediate since it is given by left multiplication on the image $f(\mathbf{1}) \in I^R$ in both cases.

Since in quasi-Frobenius algebras the A-duals of simple right A-modules are simple left A-modules [5, p. 396], the direct sum decomposition (3.23a) into simple submodules is induced by the corresponding decomposition (2.22) of A^R_A.

${}_A\overrightarrow{I^R} = {}_A\hat{I}^L$ follows since using (3.14a and b), (1.10), and (1.8), the left A-module structures of ${}_A\hat{I}^L$ and ${}_A I^R$ are related as required by (2.6),

$$\begin{aligned}\langle a\star\lambda, r\rangle &:= \big\langle a^{(1)}\cdot\lambda\cdot S\big(a^{(2)}\big), r\big\rangle = \big\langle S^2\big(a^{(2)}\big)\rightharpoonup\lambda\leftharpoonup S^{-1}\big(a^{(1)}\big), r\big\rangle\\ &= \big\langle\lambda, S^{-1}\big(a^{(1)}\big)rS^2\big(a^{(2)}\big)\big\rangle = \big\langle\lambda, S^{-1}\big(a^{(1)}\big)r\Pi^R\big(S^2\big(a^{(2)}\big)\big)\big\rangle\\ &= \big\langle\lambda, S^{-1}\big(a^{(1)}\big)rS^2\big(\Pi^R\big(a^{(2)}\big)\big)\big\rangle = \big\langle\lambda, S^{-1}\big(a^{(1)}\big)rS^3\big(\overline{\Pi}^L\big(a^{(2)}\big)\big)\big\rangle\\ &= \big\langle\lambda, S^{-1}\big(\mathbf{1}^{(1)}\big)S^{-1}(a)rS^3\big(\mathbf{1}^{(2)}\big)\big\rangle = \big\langle\mathbf{1}\star\lambda, S^{-1}(a)r\big\rangle\\ &= \big\langle\lambda, S^{-1}(a)r\big\rangle, \quad a\in A, \lambda\in\hat{I}^L,\ r\in I^R,\end{aligned} \tag{3.27}$$

and the restriction of the canonical pairing to these integrals is non-degenerate. Hence, ${}_A\hat{I}^L$ is also semisimple and the decomposition (3.23b) follows because z_p^R is a central idempotent in A and $I_p^R := z_p^R I^R = I^R z_p^R = I^R S^{-1}(z_p^R) = S^{-1}(z_p^R) I^R$. □

Corollary 3.5. *$\hat{I}^L$ becomes a free rank one left A^R- and A^L-module by restricting its left A-module structure ${}_A\hat{I}^L \equiv ({}_A\hat{I}^L, \star)$ to these canonical subalgebras.*

Proof. Due to Corollary 3.4, ${}_A\hat{I}^L$ is the right conjugate of ${}_A I^R$, that is ${}_A I^R$ is the left conjugate of ${}_A\hat{I}^L$, because $\overleftarrow{\overrightarrow{M}} = M$ for any left A-module M. Hence, if we prove the left A-module isomorphism $A^L \simeq \hat{I}^L \times I^R$ then Lemma 2.8 and Corollary 2.7 lead to the desired result.

The restriction of the multiple WHM isomorphism ${}_A\widehat{A}_A^A \simeq {}_A(\hat{I}^L \times A_A^A)$ in (3.15) to the right invariants leads to the isomorphism ${}_A I(\widehat{A}_A) \simeq {}_A I(\hat{I}^L \times A_A)$ of left A-modules, where the left A-module structure of the right invariants is inherited from that of the corresponding multiple WHM. In our case, $I(\widehat{A}_A) \equiv I(\widehat{A}_A, \mu_R) = \widehat{A}^L$ and $I(\hat{I}^L \times A_A) = \hat{I}^L \times I^R$. The latter equality can be seen by using the form (3.7b) of the projection P_A to right invariants of the WHM $\hat{I}^L \times A_A^A$. To prove the former equality we note that the invariants of the right A-module $(\widehat{A}_A, \mu_R)$ are the coinvariants of the dual left $\widehat{A}$-comodule $({}^{\widehat{A}}\widehat{A}, \hat{\delta}_L)$ given by (3.5a). Since in this case $\delta_L(\varphi) = \widehat{S}(\varphi^{(2)}) \otimes \varphi^{(1)}$, applying $\widehat{S}^{-1} \otimes \varepsilon$ to the defining identity (3.4) of left coinvariants and using (1.10) one arrives at $\widehat{A}^L = C({}^{\widehat{A}}\widehat{A}, \hat{\delta}_L) = I(\widehat{A}_A, \mu_R)$. Therefore, $\widehat{A}^L \simeq \hat{I}^L \times I^R$ as left A-modules. However, $\widehat{A}^L \simeq A^L$ also holds since the invertible map $\widehat{S} \circ \kappa_L : A^L \to \widehat{A}^L$ with κ_L in (1.5) is an A-module map

$$\begin{aligned}\big(\widehat{S}\circ\kappa_L\big)\big(a\cdot x^L\big) &:= \big(\widehat{S}\circ\kappa_L\big)\big(\Pi^L\big(ax^L\big)\big) = \widehat{S}\big(\Pi^L\big(ax^L\big)\rightharpoonup\hat{\mathbf{1}}\big) = \widehat{S}\big(ax^L\rightharpoonup\hat{\mathbf{1}}\big)\\ &= \widehat{S}\big(a\rightharpoonup\kappa_L\big(x^L\big)\big) = \widehat{S}\big(\kappa_L\big(x^L\big)\big)\leftharpoonup S^{-1}(a)\\ &=: a\cdot\big(\widehat{S}\circ\kappa_L\big)\big(x^L\big),\end{aligned} \tag{3.28}$$

where $a \in A$ and $x^L \in A^L$. Thus, $A^L \simeq \hat{I}^L \times I^R$ as left A-modules. □

4. Existence of non-degenerate left integrals in weak Hopf algebras

Here we prove the generalization of the Larson–Sweedler theorem [10].

Theorem 4.1. *A finite dimensional weak bialgebra A over a field k is a weak Hopf algebra iff there exists a non-degenerate left integral in A.*

Proof. *Sufficiency*. A left integral $l \in A$ obeys the defining property $al = \Pi^L(a)l$, $a \in A$. Non-degeneracy means that the maps

$$R_l : \widehat{A} \to A, \quad \varphi \mapsto (\varphi \rightharpoonup l), \qquad L_l : \widehat{A} \to A, \quad \varphi \mapsto (l \leftharpoonup \varphi)$$

are bijections. This implies that there exist $\lambda, \rho \in \widehat{A}$ such that $l \leftharpoonup \rho \equiv L_l(\rho) = \mathbf{1} = R_l(\lambda) \equiv \lambda \rightharpoonup l$. Let us define the k-linear maps $S : A \to A$ and $\widehat{S} : \widehat{A} \to \widehat{A}$ by

$$\begin{aligned} S(a) &:= \big(R_l \circ \widehat{L}_\lambda\big)(a) \equiv (\lambda \leftharpoonup a) \rightharpoonup l = l^{(1)}\big\langle a l^{(2)}, \lambda\big\rangle, \\ \widehat{S}(\psi) &:= \big(\widehat{R}_\lambda \circ L_l\big)(\psi) \equiv (l \leftharpoonup \psi) \rightharpoonup \lambda = \lambda^{(1)}\big\langle \psi\lambda^{(2)}, l\big\rangle. \end{aligned} \tag{4.1}$$

They are transposed to each other with respect to the canonical pairing and $\widehat{S}(\rho) = \lambda$. Now we prove, that λ (ρ) is a non-degenerate left (right) integral in $\widehat{A}$ obeying $l \rightharpoonup \lambda = \hat{\mathbf{1}} = l \rightharpoonup \rho$.

Since R_l and L_l are bijections, the identities

$$\begin{aligned} R_l(\psi\lambda) &= (\psi\lambda) \rightharpoonup l = \psi \rightharpoonup (\lambda \rightharpoonup l) = \psi \rightharpoonup \mathbf{1} = \widehat{\Pi}^L(\psi) \rightharpoonup \mathbf{1} = \widehat{\Pi}^L(\psi) \rightharpoonup (\lambda \rightharpoonup l) \\ &= R_l\big(\widehat{\Pi}^L(\psi)\lambda\big), \\ L_l(\rho\psi) &= l \leftharpoonup (\rho\psi) = (l \leftharpoonup \rho) \leftharpoonup \psi = \mathbf{1} \leftharpoonup \psi = \mathbf{1} \leftharpoonup \widehat{\Pi}^R(\psi) = (l \leftharpoonup \rho) \leftharpoonup \widehat{\Pi}^R(\psi) \\ &= L_l\big(\rho\widehat{\Pi}^R(\psi)\big) \end{aligned} \tag{4.2}$$

imply that λ (ρ) is a left (right) integral in $\widehat{A}$. Using the properties $l \leftharpoonup \rho = \mathbf{1} = \lambda \rightharpoonup l$,

$$\begin{aligned} \widehat{\Pi}^R(l \rightharpoonup \rho) &= \widehat{\Pi}^R\big(\rho^{(1)}\big)\big\langle \rho^{(2)}, l\big\rangle = \hat{\mathbf{1}}^{(1)}\big\langle \rho\hat{\mathbf{1}}^{(2)}, l\big\rangle = \hat{\mathbf{1}}^{(1)}\big\langle \hat{\mathbf{1}}^{(2)}, l \leftharpoonup \rho\big\rangle = \hat{\mathbf{1}}, \\ \widehat{\widehat{\Pi}}^R(l \rightharpoonup \lambda) &= \widehat{\widehat{\Pi}}^R\big(\lambda^{(1)}\big)\big\langle \lambda^{(2)}, l\big\rangle = \hat{\mathbf{1}}^{(1)}\big\langle \hat{\mathbf{1}}^{(2)}\lambda, l\big\rangle = \hat{\mathbf{1}}^{(1)}\big\langle \hat{\mathbf{1}}^{(2)}, \lambda \rightharpoonup l\big\rangle = \hat{\mathbf{1}}, \end{aligned} \tag{4.3}$$

which imply that $l \rightharpoonup \rho = \hat{\mathbf{1}} = l \rightharpoonup \lambda$ since $l \rightharpoonup \rho, l \rightharpoonup \lambda \in \widehat{A}^L$ and both of the $\widehat{A}^L - A^R$ and $\widehat{A}^L - A^L$ pairings are nondegenerate. The proved properties of $\lambda, \rho \in \widehat{A}$ allow us to construct the inverse of the map $\widehat{S}$:

$$\widehat{S}^{-1}(\psi) := \big(\widehat{R}_\rho \circ R_l\big)(\psi) \equiv (\psi \rightharpoonup l) \rightharpoonup \rho = \rho^{(1)}\big\langle \rho^{(2)}\psi, l\big\rangle. \tag{4.4}$$

Indeed, for all $\psi \in \widehat{A}$ one obtains

$$\begin{aligned} \big(\widehat{S}^{-1} \circ \widehat{S}\big)(\psi) &:= \rho^{(1)}\big\langle \rho^{(2)}\lambda^{(1)}, l\big\rangle\big\langle \psi\lambda^{(2)}, l\big\rangle = \rho^{(1)}\big\langle \rho^{(2)}\widehat{\Pi}^R\big(\psi^{(1)}\big)\lambda^{(1)}, l\big\rangle\big\langle \psi^{(2)}\lambda^{(2)}, l\big\rangle \\ &= \rho^{(1)}\psi^{(1)}\big\langle \rho^{(2)}\psi^{(2)}\lambda^{(1)}, l\big\rangle\big\langle \psi^{(3)}\lambda^{(2)}, l\big\rangle \\ &= \rho^{(1)}\psi^{(1)}\big\langle \rho^{(2)}\widehat{\Pi}^L\big(\psi^{(2)}\big)\lambda^{(1)}, l\big\rangle\big\langle \lambda^{(2)}, l\big\rangle \\ &= \rho^{(1)}\psi^{(1)}\big\langle \rho^{(2)}\widehat{\Pi}^L\big(\psi^{(2)}\big), l\big\rangle = \rho^{(1)}\psi\big\langle \rho^{(2)}, l\big\rangle = \psi. \end{aligned} \tag{4.5}$$

Therefore, the transposed map $S^{-1} := (\widehat{S}^{-1})^t \equiv L_l \circ \widehat{L}_\rho$ is the inverse of S and invertibility of $S := R_l \circ \widehat{L}_\lambda$ given in (4.1) implies that $\widehat{L}_\lambda$ and $\widehat{L}_\rho$ are invertible. Hence, λ and ρ are non-degenerate left and right integrals in $\widehat{A}$, respectively.

Since $\rho \in \widehat{A}$ is a non-degenerate right integral, there exists $r \in A$ such that $\rho \leftharpoonup r = \hat{\mathbf{1}}$. In a similar way as before, one proves that r is a right integral obeying $r \leftharpoonup \rho = \mathbf{1}$:

$$\widehat{L}_\rho(ra) = \rho \leftharpoonup (ra) = \hat{\mathbf{1}} \leftharpoonup a = \hat{\mathbf{1}} \leftharpoonup \Pi^R(a) = \widehat{L}_\rho\big(r\Pi^R(a)\big),$$
$$\Pi^R(r \leftharpoonup \rho) = \big\langle \rho, r^{(1)}\big\rangle S\big(\overline{\Pi}^L\big(r^{(2)}\big)\big) = \big\langle \rho, r\mathbf{1}^{(1)}\big\rangle S\big(\mathbf{1}^{(2)}\big) = \big\langle \rho \leftharpoonup r, \mathbf{1}^{(1)}\big\rangle S\big(\mathbf{1}^{(2)}\big) = \mathbf{1}, \quad (4.6)$$

hence, $r \leftharpoonup \rho = \mathbf{1}$ follows since ρ is a right integral and the $A^R - \widehat{A}^R$ pairing is non-degenerate. But then $S = L_r \circ \widehat{R}_\rho$ also holds (therefore, r is non-degenerate), because

$$\begin{aligned}\big(S^{-1} \circ L_r \circ \widehat{R}_\rho\big)(a) &= \big(L_l \circ \widehat{L}_\rho \circ L_r \circ \widehat{R}_\rho\big)(a) = \big\langle r^{(1)}a, \rho\big\rangle\big\langle r^{(2)}l^{(1)}, \rho\big\rangle l^{(2)} \\ &= \big\langle r^{(1)}a^{(1)}, \rho\big\rangle\big\langle r^{(2)}\Pi^L\big(a^{(2)}\big)l^{(1)}, \rho\big\rangle l^{(2)} \\ &= \big\langle r^{(1)}a^{(1)}, \rho\big\rangle\big\langle r^{(2)}a^{(2)}l^{(1)}, \rho\big\rangle a^{(3)}l^{(2)} \\ &= \big\langle r^{(1)}, \rho\big\rangle\big\langle r^{(2)}\Pi^R\big(a^{(1)}\big)l^{(1)}, \rho\big\rangle a^{(2)}l^{(2)} = \big\langle \Pi^R\big(a^{(1)}\big)l^{(1)}, \rho\big\rangle a^{(2)}l^{(2)} \\ &= \big\langle l^{(1)}, \rho\big\rangle al^{(2)} = a. \end{aligned} \quad (4.7)$$

Now, the defining properties (1.2) of the antipode are fulfilled for the map $S := R_l \circ \widehat{L}_\lambda = L_r \circ \widehat{R}_\rho$, because for all $a \in A$, one has

$$a^{(1)}S\big(a^{(2)}\big) = a^{(1)}l^{(1)}\big\langle a^{(2)}l^{(2)}, \lambda\big\rangle = \Pi^L(a)l^{(1)}\big\langle l^{(2)}, \lambda\big\rangle = \Pi^L(a), \quad (4.8a)$$
$$S\big(a^{(1)}\big)a^{(2)} = r^{(2)}a^{(2)}\big\langle r^{(1)}a^{(1)}, \rho\big\rangle = r^{(2)}\Pi^R(a)\big\langle r^{(1)}, \rho\big\rangle = \Pi^R(a), \quad (4.8b)$$
$$\begin{aligned}S\big(a^{(1)}\big)a^{(2)}S\big(a^{(3)}\big) &= \Pi^R\big(a^{(1)}\big)S\big(a^{(2)}\big) = \Pi^R\big(a^{(1)}\big)l^{(1)}\big\langle a^{(2)}l^{(2)}, \lambda\big\rangle \\ &= l^{(1)}\big\langle al^{(2)}, \lambda\big\rangle = S(a). \end{aligned} \quad (4.8c)$$

Necessity. The statement follows from Lemma 2.6 and results [2, Theorem 3.16], but for completeness we give a full proof using the results of the previous chapter.

Applying the structure theorem of multiple WHMs to ${}_A\widehat{A}^A_A$ given by (3.14), we get the isomorphism ${}_A\widehat{A}^A_A \simeq {}_A(\hat{I}^L \times A^A_A)$. Moreover, the restriction of the left A-module structure of ${}_A\hat{I}^L \equiv ({}_A\hat{I}^L, \star)$ to the coideal subalgebra $A^R \subset A$ leads to a free A^R-module ${}_{A^R}\hat{I}^L \equiv ({}_{A^R}\hat{I}^L, \star)$ with a single generator $\lambda_0 \in \hat{I}^L$ due to Corollary 3.5. Hence, using the multiple WHM isomorphism $V : \hat{I}^L \times A \to \widehat{A}$ given in (3.19) and the presence of the separating idempotent in $\hat{I}^L \times A := \mathbf{1}^{(1)} \star \hat{I}^L \otimes \mathbf{1}^{(2)} \cdot A = \hat{I}^L \cdot S(\mathbf{1}^{(1)}) \otimes \mathbf{1}^{(2)}A$, which follows from (3.11), (1.4), and from the property $\Delta(\mathbf{1}) \in A^R \otimes A^L$, one obtains

$$\begin{aligned}\widehat{A} &= V\big(\hat{I}^L \times A\big) = V\big(\big(A^R \star \lambda_0\big) \times A\big) = V\big(\big(\lambda_0 \cdot S\big(A^R\big)\big) \times A\big) = V\big(\lambda_0 \times S\big(A^R\big)A\big) \\ &= V(\lambda_0 \times A) := \lambda_0 \cdot A := S(A) \rightharpoonup \lambda_0 = A \rightharpoonup \lambda_0, \end{aligned} \quad (4.9)$$

which implies the non-degeneracy of the left integral λ_0 in the weak Hopf algebra $\widehat{A}$. $\square$

Since a non-degenerate left integral in a WHA provides a non-degenerate associative bilinear form on the dual WHA:

Corollary 4.2. *A finite dimensional weak Hopf algebra is a Frobenius algebra.*

5. Grouplike elements and invertible modules

In this chapter first we define (left/right) grouplike elements in a WHA A. Then we give two equivalent descriptions of invertible A-modules in terms of the canonical coideal subalgebras in A and in terms of left (right) grouplike elements in the dual WHA $\widehat{A}$.

The set of *grouplike elements* $G(H)$ in a Hopf algebra H can be defined to be [17] $G(H) := \{g \in H \mid \Delta(g) = g \otimes g,\ \varepsilon(g) \neq 0\}$. The grouplike elements are linearly independent, they obey the property $S(g)g = \mathbf{1}$ and they form a group. The generalization of this notion to a weak Hopf algebra A

$$G(A) := \bigl\{g \in A \bigm| \Delta(\mathbf{1})(g \otimes g) = \Delta(g) = (g \otimes g)\Delta(\mathbf{1}), gS(g) = \mathbf{1}\bigr\}$$

given in [2, p. 433] seems to be too restrictive, hence we introduce slightly softened generalizations as well.

Definition 5.1. The set of right/left grouplike elements $G_{R/L}(A)$ in a weak Hopf algebra A is defined to be

$$\begin{aligned} G_R(A) := \bigl\{g \in A \bigm| (g \otimes g)\Delta(\mathbf{1}) = \Delta(g) = \Delta(\mathbf{1})\bigl(g \otimes \Pi^L(g)^{-1}g\bigr),\\ \Pi^{R/L}(g) \in A_*^{R/L}\bigr\}, \end{aligned} \tag{5.1a}$$

$$\begin{aligned} G_L(A) := \bigl\{g \in A \bigm| \bigl(g\Pi^R(g)^{-1} \otimes g\bigr)\Delta(\mathbf{1}) = \Delta(g) = \Delta(\mathbf{1})(g \otimes g),\\ \Pi^{R/L}(g) \in A_*^{R/L}\bigr\}, \end{aligned} \tag{5.1b}$$

where $A_*^{R/L}$ denote the set of invertible elements in $A^{R/L}$. The set of grouplike elements in A is defined to be the intersection $G(A) := G_R(A) \cap G_L(A)$.

Using the form (1.11) of the maps $\Pi^{R/L}$, the defining properties (5.1) lead to the relations

$$\begin{aligned} g \in G_R(A)\colon\quad & \Pi^R(g) = S(g)\Pi^L(g)^{-1}g, \quad \Pi^L(g) = gS(g)\\ & \Rightarrow \quad g \in A_*,\ \Pi^R(g) = \mathbf{1}, \end{aligned} \tag{5.2a}$$

$$\begin{aligned} g \in G_L(A)\colon\quad & \Pi^L(g) = g\Pi^R(g)^{-1}S(g), \quad \Pi^R(g) = S(g)g\\ & \Rightarrow \quad g \in A_*,\ \Pi^L(g) = \mathbf{1}, \end{aligned} \tag{5.2b}$$

that is elements of $G_{R/L}(A)$ are themselves invertible. Using (5.1), (5.2) it is easy to show that $G_R(A)$ and $G_L(A)$, hence $G(A)$, too, are groups, $G_L(A) = S(G_R(A))$, and the definition of grouplike elements $G(A)$ above is equivalent to that of given in [2]. For example, $gh \in G_R(A)$ if $g, h \in G_R(A)$ since

$$\begin{aligned}
\Pi^R(gh) &= \Pi^R\bigl(\Pi^R(g)h\bigr) = \Pi^R(\mathbf{1}h) = \mathbf{1} \in A_*^R,\\
\Pi^L(gh) &= \Pi^L\bigl(g\Pi^L(h)\bigr) = g^{(1)}\Pi^L(h)S\bigl(g^{(2)}\bigr) = ghS(h)S(g) \in A_*^L,\\
\Delta(gh) &= (g\otimes g)\Delta(\mathbf{1})\Delta(h) = (g\otimes g)\Delta(h) = (gh\otimes gh)\Delta(\mathbf{1}),\\
\Delta(gh) &= \Delta(g)\Delta(\mathbf{1})\bigl(h\otimes \Pi^L(h)^{-1}h\bigr) = \Delta(g)\bigl(h\otimes S\bigl(h^{-1}\bigr)\bigr)\\
&= \Delta(\mathbf{1})\bigl(g\otimes S\bigl(g^{-1}\bigr)\bigr)\bigl(h\otimes S\bigl(h^{-1}\bigr)\bigr) = \Delta(\mathbf{1})\bigl(gh\otimes \Pi^L(gh)^{-1}gh\bigr).
\end{aligned}$$

Corollary 5.2. *The element $g \in A$ is right* (*left*) *grouplike iff g is invertible and obeys the property $\Delta(g) = (g\otimes g)\Delta(\mathbf{1})$ $(\Delta(g) = \Delta(\mathbf{1})(g\otimes g))$.*

Proof. If $g \in A$ is right (left) grouplike it is invertible due to the discussion above, while the required coproduct property follows by definition. Conversely, the relations

$$\begin{aligned}
\mathbf{1} = g^{-1}g = g^{-1}(\mathrm{id}\otimes\varepsilon)\bigl(\Delta(g)\bigr) = \mathbf{1}^{(1)}\varepsilon\bigl(g\mathbf{1}^{(2)}\bigr) =: \Pi^R(g),\\
\Pi^L(g) = g^{(1)}S\bigl(g^{(2)}\bigr) = g\mathbf{1}^{(1)}S\bigl(\mathbf{1}^{(2)}\bigr)S(g) = gS(g),
\end{aligned}$$

imply that $\Pi^{R/L}(g) \in A_*^{R/L}$. In conclusion, using (1.8) one derives

$$\mathbf{1}^{(1)}g\otimes \mathbf{1}^{(2)} = g^{(1)}\otimes \Pi^L\bigl(g^{(2)}\bigr) = g\mathbf{1}^{(1)}\otimes \Pi^L\bigl(g\mathbf{1}^{(2)}\bigr) = g\mathbf{1}^{(1)}\otimes g\mathbf{1}^{(2)}S(g).$$

Multiplying this identity by $\mathbf{1}\otimes S(g^{-1})$ from the right and using the form of $\Pi^L(g)$, one arrives the other coproduct property of a right grouplike element in (5.1a). The proof for left grouplike elements is similar. □

We note that the set $G(A)$ in $G_R(A)$ can also be given by the subset of elements satisfying $\Pi^L(g) = \mathbf{1}$ or by the subset of pointwise invariant elements with respect to S^2. For verification of the latter claim, we note that if $g = S^2(g)$ holds for $g \in G_R(A)$ then $\Pi^L(g) = gS(g) = S^2(g)S(g) = S(\Pi^L(g))$, that is $\Pi^L(g)$, hence $\Pi^L(g)^{-1}$, too, are in $A^L \cap A^R \subset \mathrm{Center}\, A^L$. Using (5.2a), (5.1a), and these consequences, one obtains

$$\mathbf{1} = \Pi^R(g) = S\bigl(\mathbf{1}^{(1)}\bigr)S(g)g\mathbf{1}^{(2)} \quad\Rightarrow\quad \Pi^L\bigl(g^{-1}\bigr)^{-1} = \mathbf{1} \quad\Rightarrow\quad g^{-1}, g \in G(A).$$

Now we turn to characterization of invertible modules of WHAs.

Definition 5.3. An object M of a monoidal category $(\mathcal{L}; \times, E)$ is invertible if there exists an inverse object $\overline{M} \in \operatorname{Obj}\mathcal{L}$ obeying $M \times \overline{M} \simeq E \simeq \overline{M} \times M$, where $E \in \operatorname{Obj}\mathcal{L}$ is the monoidal unit of the category and $\simeq$ denotes equivalence of objects in $\mathcal{L}$.[3]

Proposition 5.4.

(i) *Let $\mathcal{L}$ be the autonomous monoidal category of finite dimensional left A-modules of a WHA A given in Proposition* 2.2. *The module ${}_AM \in \operatorname{Obj}\mathcal{L}$ is invertible iff it becomes a free rank one left A^L- and A^R-module by restricting A to the subalgebras A^L and A^R, respectively.*

(ii) *An invertible module ${}_AM \in \operatorname{Obj}\mathcal{L}$ is semisimple. Namely, it is the direct sum of simple submodules*

$$ {}_AM = \bigoplus_p M_{(p,\tau_M(p))}, \qquad M_{(p,\tau_M(p))} := z_p^L \cdot M = z^R_{\tau_M(p)} \cdot M, $$

where $\{z_p^L\}_p \subset Z^L$ and $\{z_p^R := S(z_p^L)\}_p \subset Z^R$ are the sets of primitive orthogonal idempotents and τ_M is a permutation on them.

Proof. (i) First, we show that ${}_AM$ is invertible iff

$$ M \times \overleftarrow{M} \simeq A^L \simeq \overrightarrow{M} \times M \tag{5.3} $$

as left A-modules, where A^L is the unit left A-module given in (2.1).

If (5.3) holds then, using the natural equivalences X^L and X^R given in (2.3),

$$ \overleftarrow{M} \simeq A^L \times \overleftarrow{M} \simeq \overrightarrow{M} \times M \times \overleftarrow{M} \simeq \overrightarrow{M} \times A^L \simeq \overrightarrow{M} $$

follows, hence, M is invertible. Conversely, let $\overline{M}$ be the inverse of M and let $\sigma : \overline{M} \times M \to A^L$ and $\tau : M \times \overline{M} \to A^L$ be the corresponding invertible arrows. We will show that $\overleftarrow{M} \simeq \overline{M} \simeq \overrightarrow{M}$, which imply (5.3). The arrow

$$ \omega := \big(X^L_{A^L}\big)^{-1} (\tau \times \tau)\big(1_M \times \sigma^{-1} \times 1_{\overline{M}}\big)\big(X^R_M \times 1_{\overline{M}}\big)\tau^{-1} \in \operatorname{End}_A A^L \tag{5.4} $$

is invertible; therefore, it is given by the action of an invertible element $z^L \in Z^L := A^L \cap \operatorname{Center} A$ due to $\operatorname{End}_A A^L = Z^L \cdot$ [2, p. 402]. Hence, if $z^L_N : N \to N$ denotes the arrow given by the action $z^L \in Z^L$ for $N \in \operatorname{Obj}\mathcal{L}$ then $\{z^L_N\}_N$ is a natural automorphism of the identity functor on $\mathcal{L}$ and $\omega = z^L_{A^L}$. Defining

$$ \tilde{\tau} := \big(z^L_{A^L}\big)^{-1} \tau : M \times \overline{M} \to A^L, $$

[3] In case of symmetric or braided monoidal categories invertibility is defined by the condition $E \simeq M \times \overline{M}$ [7].

we have

$$\tilde{\tau} = \tau\big(z^L_{M\times\overline{M}}\big)^{-1} = \tau\big(\big(z^L_M\big)^{-1} \times 1_{\overline{M}}\big)$$

due to naturality and (1.4). Therefore,

$$\big(z^L_M\big)^{-1} \times 1_{\overline{M}} = \tau^{-1}\tilde{\tau} = \tau^{-1}\omega^{-1}\tau,$$

which leads to

$$z^L_M = \big(X^L_M\big)^{-1}(\tau \times 1_M)\big(1_M \times \sigma^{-1}\big)X^R_M$$

due to the form (5.4) of ω and faithfulness of $- \times 1_{\overline{M}}$. Hence, using naturality and (1.4)

$$\begin{aligned} 1_M &= \big(z^L_M\big)^{-1} z^L_M = \big(z^L_M\big)^{-1}\big(X^L_M\big)^{-1}(\tau \times 1_M)\big(1_M \times \sigma^{-1}\big)X^R_M \\ &= \big(X^L_M\big)^{-1}(\tilde{\tau} \times 1_M)\big(1_M \times \sigma^{-1}\big)X^R_M. \end{aligned} \tag{5.5a}$$

Then

$$\big(X^R_{\overline{M}}\big)^{-1}\big(1_{\overline{M}} \times \tilde{\tau}\big)\big(\sigma^{-1} \times 1_{\overline{M}}\big)X^L_{\overline{M}} = 1_{\overline{M}} \tag{5.5b}$$

also holds because of faithfulness of $1_M \times -$ and because of the identity

$$\begin{aligned} 1_{A^L} &= \tilde{\tau}\tilde{\tau}^{-1} = \tilde{\tau}\big(\big(X^L_M\big)^{-1} \times 1_{\overline{M}}\big)\big(\tilde{\tau} \times 1_M \times 1_{\overline{M}}\big)\big(1_M \times \sigma^{-1} \times 1_{\overline{M}}\big)\big(X^R_M \times 1_{\overline{M}}\big)\tilde{\tau}^{-1} \\ &= \big(X^L_{A^L}\big)^{-1}(\tilde{\tau} \times \tilde{\tau})\big(1_M \times \sigma^{-1} \times 1_{\overline{M}}\big)\big(X^R_M \times 1_{\overline{M}}\big)\tilde{\tau}^{-1} \\ &= \tilde{\tau}\Big[1_M \times \big(x^r_{\overline{M}}\big)^{-1}\big(1_{\overline{M}} \times \tilde{\tau}\big)\big(\sigma^{-1} \times 1_{\overline{M}}\big)x^l_{\overline{M}}\Big]\tilde{\tau}^{-1}. \end{aligned} \tag{5.6}$$

Thus, using the right and left evaluation maps defined in (2.8) and (2.14), respectively,

$$\overleftarrow{\mu} := \big(X^L_{\overline{M}}\big)^{-1}\big(E^l_M \times 1_{\overline{M}}\big)\Big(1_{\overleftarrow{M}} \times \tilde{\tau}^{-1}\Big)X^R_{\overleftarrow{M}} : \overleftarrow{M} \to \overline{M}, \tag{5.7a}$$

$$\overrightarrow{\mu} := \big(X^R_{\overline{M}}\big)^{-1}\big(1_{\overline{M}} \times E^r_M\big)\Big(\sigma^{-1} \times 1_{\overrightarrow{M}}\Big)X^L_{\overrightarrow{M}} : \overrightarrow{M} \to \overline{M} \tag{5.7b}$$

provide the equivalences $\overleftarrow{M} \simeq \overline{M} \simeq \overrightarrow{M}$ with the inverse arrows

$$\overleftarrow{\mu}^{-1} = \big(x^l_{\overleftarrow{M}}\big)^{-1}\big(\sigma \times 1_{\overleftarrow{M}}\big)\big(1_{\overline{M}} \times C^l_M\big)X^R_{\overline{M}} : \overline{M} \to \overleftarrow{M}, \tag{5.8a}$$

$$\overrightarrow{\mu}^{-1} = \big(X^R_{\overrightarrow{M}}\big)^{-1}\big(1_{\overrightarrow{M}} \times \tilde{\tau}\big)\big(C^r_M \times 1_{\overline{M}}\big)X^L_{\overline{M}} : \overline{M} \to \overrightarrow{M} \tag{5.8b}$$

due to the rigidity identities (2.10) and (2.15), respectively, and due to (5.5a and b). For example,

$$\begin{aligned}\overleftarrow{\mu}^{-1}\overleftarrow{\mu} &:= \Big[\big(X^L_{\overleftarrow{M}}\big)^{-1}\big(\sigma \times 1_{\overleftarrow{M}}\big)\big(1_{\overline{M}} \times C^l_M\big)X^R_{\overline{M}}\Big]\Big[\big(X^L_{\overline{M}}\big)^{-1}\big(E^l_M \times 1_{\overline{M}}\big)\big(1_{\overleftarrow{M}} \times \tilde{\tau}^{-1}\big)X^R_{\overleftarrow{M}}\Big] \\ &= \big(X^L_{\overleftarrow{M}}\big)^{-1}\big(E^l_M \times 1_{\overleftarrow{M}}\big)\Big(1_{\overleftarrow{M}} \times \big[\big(X^R_M\big)^{-1}(1_M \times \sigma)\big(\tilde{\tau}^{-1} \times 1_M\big)X^L_M\big] \times 1_{\overleftarrow{M}}\Big) \\ &\quad \big(1_{\overleftarrow{M}} \times C^l_M\big)X^R_{\overleftarrow{M}} = \big(X^L_{\overleftarrow{M}}\big)^{-1}\big(E^l_M \times 1_{\overleftarrow{M}}\big)\big(1_{\overleftarrow{M}} \times C^l_M\big)X^R_{\overleftarrow{M}} = 1_{\overleftarrow{M}},\end{aligned}$$

where we used the inverse of (5.5a) in the third equality and (2.10b) in the fourth one.

Now we prove that (5.3) is fulfilled iff M becomes a free rank one A^L- and A^R-module by restricting the left A-action to these subalgebras. If (5.3) holds then the statement follows from Lemma 2.8 and Corollary 2.7. Conversely, suppose that ${}_AM$ becomes a free A^L- and A^R-module with a single generator $m \in M$ by restricting the A-action to these subalgebras. The elements $\widehat{m}_l$ and $\widehat{m}_r$ of the k-dual $\widehat{M}$ of M defined by

$$\big\langle \widehat{m}_l, x^R \cdot m\big\rangle_M := \varepsilon\big(x^R\big), \qquad \big\langle \widehat{m}_r, x^L \cdot m\big\rangle_M := \varepsilon\big(x^L\big), \quad x^{R/L} \in A^{R/L} \tag{5.9}$$

are A^L- and A^R-generators of $\overleftarrow{M}$ and $\overrightarrow{M}$, respectively, because the counit ε is a non-degenerate functional on A^R and on A^L. Moreover, choosing dual bases $\{e_i\}_i$, $\{f_i\}_i$ in A^L with respect to the counit ε, the bases $\{e_i \cdot \widehat{m}_l\}_i \subset \overleftarrow{M}$, $\{S^{-1}(f_i) \cdot m\}_i \subset M$, and $\{S^{-1}(f_i) \cdot \widehat{m}_r\}_i \subset \overrightarrow{M}$, $\{e_i \cdot m\}_i \subset M$ become dual to each other. Indeed, for the dual A^L-bases we have

$$\delta_{ij} = \varepsilon(e_i f_j) = \varepsilon\big(f_j S^2(e_i)\big) = \varepsilon\big(S(e_i)S^{-1}(f_j)\big). \tag{5.10}$$

The third equality follows from the invariance of the counit with respect to the antipode: $\varepsilon = \varepsilon \circ S$. The second is the consequence of the identities (1.14), (1.15) claiming that S^2 is the Nakayama automorphism $\theta_L : A^L \to A^L$ corresponding to the counit as a non-degenerate functional on A^L. Therefore,

$$\begin{aligned}\delta_{ij} &= \varepsilon\big(S(e_i)S^{-1}(f_j)\big) := \big\langle \widehat{m}_l, S(e_i)S^{-1}(f_j) \cdot m\big\rangle_M = \big\langle e_i \cdot \widehat{m}_l, S^{-1}(f_j) \cdot m\big\rangle_M, \\ \delta_{ij} &= \varepsilon\big(S^{-2}(f_j)e_i\big) := \big\langle \widehat{m}_r, S^{-2}(f_j)e_i \cdot m\big\rangle_M = \big\langle S^{-1}(f_j) \cdot \widehat{m}_r, e_i \cdot m\big\rangle_M.\end{aligned} \tag{5.11}$$

Thus, we can prove that the left and right coevaluation maps $C^l_M : A^L \to M \times \overleftarrow{M}$ and $C^r_M : A^L \to \overrightarrow{M} \times M$ defined in (2.8) and (2.14) are invertible, that is (5.3) holds: using that $\mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)} = S^{-1}(f_i) \otimes e_i$ (summation suppressed), rank one A^L- and A^R-freeness of M in the fourth equalities, respectively, and (2.13a) in the sixth equality of (5.12b), one obtains

$$\begin{aligned}M \times \overleftarrow{M} &:= \mathbf{1}^{(1)} \cdot M \otimes \mathbf{1}^{(2)} \cdot \overleftarrow{M} = \mathbf{1}^{(1)} \cdot M \otimes \mathbf{1}^{(2)} A^L \cdot \widehat{m}_l \\ &= \mathbf{1}^{(1)} S^{-1}\big(A^L\big) \cdot M \otimes \mathbf{1}^{(2)} \cdot \widehat{m}_l = \mathbf{1}^{(1)} A^L \cdot m \otimes \mathbf{1}^{(2)} \cdot \widehat{m}_l \\ &= A^L S^{-1}(f_i) \cdot m \otimes e_i \cdot \widehat{m}_l = C^l_M\big(A^L\big),\end{aligned} \tag{5.12a}$$

$$\begin{aligned}\overrightarrow{M} \times M &:= \mathbf{1}^{(1)} \cdot \overrightarrow{M} \otimes \mathbf{1}^{(2)} \cdot M = \mathbf{1}^{(1)} A^R \cdot \widehat{m}_r \otimes \mathbf{1}^{(2)} \cdot M \\ &= \mathbf{1}^{(1)} \cdot \widehat{m}_r \otimes \mathbf{1}^{(2)} S\big(A^R\big) \cdot M = \mathbf{1}^{(1)} \cdot \widehat{m}_r \otimes \mathbf{1}^{(2)} A^R \cdot m \\ &= S^{-1}(f_i) \cdot \widehat{m}_r \otimes A^R e_i \cdot m = A^L S^{-1}(f_i) \cdot \widehat{m}_r \otimes e_i \cdot m = C^r_M\big(A^L\big),\end{aligned} \tag{5.12b}$$

i.e., C^l_M and C^r_M are surjective. Injectivity of C^l_M and C^r_M follow from the faithfulness of M as a left A^L- and A^R-module, respectively.

(ii) From (5.3) and Lemma 2.8 we can deduce that

$$\mathrm{End}_A M \subset \mathrm{End}_{A^R} M \cap \mathrm{End}_{A^L} M = \bigl(\mathrm{Center}\, A^L\bigr)\cdot = \bigl(\mathrm{Center}\, A^R\bigr)\cdot. \tag{5.13}$$

Let $m \in M$ be a free A^L-generator. The action by an element $x^L \in \mathrm{Center}\, A^L$ on M is an element of $\mathrm{End}_A M$ only if

$$\Pi^L(a)x^L \cdot m = a^{(1)}S\bigl(a^{(2)}\bigr)x^L \cdot m = a^{(1)}x^L S\bigl(a^{(2)}\bigr)\cdot m = \Pi^L\bigl(ax^L\bigr)\cdot m, \quad a \in A, \tag{5.14}$$

i.e., only if $\Pi^L(a)x^L = \Pi^L(ax^L)$ for all $a \in A$. However, this relation implies that $x^L \in \mathrm{Center}\, A$:

$$\begin{aligned} S(a)x^L &= S\bigl(a^{(1)}\bigr)\Pi^L\bigl(a^{(2)}\bigr)x^L = S\bigl(a^{(1)}\bigr)\Pi^L\bigl(a^{(2)}x^L\bigr) = S\bigl(a^{(1)}\bigr)a^{(2)}x^L S\bigl(a^{(3)}\bigr) \\ &= \Pi^R\bigl(a^{(1)}\bigr)x^L S\bigl(a^{(2)}\bigr) = x^L S(a), \quad a \in A. \end{aligned} \tag{5.15}$$

Therefore, $x^L \in A^L \cap \mathrm{Center}\, A =: Z^L$, that is $\mathrm{End}_A M \subset Z^L\cdot$. The opposite containment is trivial. The proof of the relation $\mathrm{End}_A M = Z^R\cdot$ is similar. Hence, the direct summands of ${}_AM$ in the statement (ii) are indecomposable submodules. Since ${}_AM$ is a free rank one A^L- and A^R-module due to (i), τ_M is a permutation and the k-dimensions of the indecomposable submodules $M_{(p,\tau_M(p))}$ saturate the lower bound (2.25) given in Lemma 2.6. Therefore, $M_{(p,\tau_M(p))}$ is simple since it cannot contain a non-trivial submodule. □

Now we turn to the characterization of invertible $\widehat{A}$-modules in terms of right (left) grouplike elements in the WHA A. First, we give the connection between (right/left) grouplike elements in A and invertible submodules of $({}_{\widehat{A}}A, \rightharpoonup)$.

Lemma 5.5. *Let A be a WHA and let $F_a := (\widehat{A} \rightharpoonup a, \rightharpoonup)$ denote the cyclic left $\widehat{A}$-submodule of ${}_{\widehat{A}}A := ({}_{\widehat{A}}A, \rightharpoonup)$ generated by $a \in A$.*

- (i) *$g \in A$ is (right/left) grouplike iff g is an element of an invertible submodule ${}_{\widehat{A}}F$ of ${}_{\widehat{A}}A$ and g obeys the normalization conditions ($\Pi^{R/L}(g) = \mathbf{1}$) $\Pi^R(g) = \mathbf{1} = \Pi^L(g)$.*
- (ii) *The cyclic submodules $F_g, F_h \subset {}_{\widehat{A}}A$ generated by (right/left) grouplike elements are in the same module isomorphism class iff $gh^{-1} \in A^T$.*
- (iii) *In any module isomorphism class of invertible submodules of ${}_{\widehat{A}}A$, there is a submodule which contains a right (left) grouplike element.*

Proof. (i) Let $g \in G_{R/L}(A)$ or $g \in G(A)$. Clearly, F_g is a submodule of ${}_{\widehat{A}}A$ that contains g satisfying the required normalization conditions. According to Proposition 5.4(i) invertibility of F_g follows if F_g becomes a free $\widehat{A}^L$- and $\widehat{A}^R$-module with the single generator g by restricting the $\widehat{A}$-action to these subalgebras. If $g \in G_R(A)$ then the identities (1.6), (1.7) and (5.1), (5.2a) lead to the relations

$$\varphi \rightharpoonup g = \mathbf{1}^{(1)} g \langle \varphi, \mathbf{1}^{(2)} S(g)^{-1} \rangle = \mathbf{1}^{(1)} g \langle S(g)^{-1} \rightharpoonup \varphi, \Pi^L(\mathbf{1}^{(2)}) \rangle$$
$$= \big(\widehat{\Pi}^L\big(S(g)^{-1} \rightharpoonup \varphi\big) \rightharpoonup \mathbf{1}\big) g = \widehat{\Pi}^L\big(S(g)^{-1} \rightharpoonup \varphi\big) \rightharpoonup g, \quad \varphi \in \widehat{A}, \tag{5.16a}$$
$$\varphi \rightharpoonup g = g\mathbf{1}^{(1)} \langle \varphi, g\mathbf{1}^{(2)} \rangle = g\mathbf{1}^{(1)} \langle \varphi \leftharpoonup g, \overline{\Pi}^L(\mathbf{1}^{(2)}) \rangle = g\mathbf{1}^{(1)} \langle \widehat{\overline{\Pi}}^R(\varphi \leftharpoonup g), \mathbf{1}^{(2)} \rangle$$
$$= g\big(\widehat{\overline{\Pi}}^R(\varphi \leftharpoonup g) \rightharpoonup \mathbf{1}\big) = \widehat{\overline{\Pi}}^R(\varphi \leftharpoonup g) \rightharpoonup g, \quad \varphi \in \widehat{A}. \tag{5.16b}$$

They imply that $F_g \subset (\widehat{A}^R \rightharpoonup g) \cap (\widehat{A}^L \rightharpoonup g) = gA^R \cap A^R g$. Moreover, if $0 = \varphi^L \rightharpoonup g = (\varphi^L \rightharpoonup \mathbf{1})g$ or $0 = \varphi^R \rightharpoonup g = g(\varphi^R \rightharpoonup \mathbf{1}) = g(S(\varphi^R) \rightharpoonup \mathbf{1})$ for certain $\varphi^{L/R} \in \widehat{A}^{L/R}$ then $\varphi^{L/R} = 0$, because g is invertible and the maps $\hat{\kappa}_L$ in (1.5) and the antipode S are bijections. Therefore, F_g is a free rank one $\widehat{A}^R$- and $\widehat{A}^L$-module for any $g \in G_R(A)$, hence for any $g \in G(A) \subset G_R(A)$, too. The case of $g \in G_L(A)$ can be proved similarly.

Conversely, let ${}_{\widehat{A}}F$ be an invertible submodule of ${}_{\widehat{A}}A$. Then F is a right coideal in A and a free left $\widehat{A}^L$- and $\widehat{A}^R$-module with a single generator $f \in F$. Thus, one can define two projections $\Phi^L_f : \widehat{A} \to \widehat{A}^L$ and $\overline{\Pi}^R_f : \widehat{A} \to \widehat{A}^R$ by requiring

$$\Phi^L_f(\varphi) \rightharpoonup f := \varphi \rightharpoonup f, \qquad \overline{\Phi}^R_f(\varphi) \rightharpoonup f := \varphi \rightharpoonup f, \tag{5.17}$$

for $\varphi \in \widehat{A}$. They are left $\widehat{A}^L$- and $\widehat{A}^R$-module maps, respectively. Since F is a right coideal in A, defining $\hat{f}_l$ and $\hat{f}_r$ in the k-dual $\widehat{F}$ of F like in (5.9) by

$$\langle \hat{f}_r, \varphi^L \rightharpoonup f \rangle_F = \langle \hat{f}_r, f^{(1)} \rangle_F \langle f^{(2)}, \varphi^L \rangle \equiv \langle f \leftharpoonup \hat{f}_r, \varphi^L \rangle := \hat{\varepsilon}(\varphi^L), \quad \varphi^L \in \widehat{A}^L,$$
$$\langle \hat{f}_l, \varphi^R \rightharpoonup f \rangle_F = \langle \hat{f}_l, f^{(1)} \rangle_F \langle f^{(2)}, \varphi^R \rangle \equiv \langle f \leftharpoonup \hat{f}_l, \varphi^R \rangle := \hat{\varepsilon}(\varphi^R), \quad \varphi^R \in \widehat{A}^R, \tag{5.18}$$

we have $\Pi^L(f \leftharpoonup \hat{f}_r) = \mathbf{1} = \Pi^R(f \leftharpoonup \hat{f}_l)$ and

$$\Phi^L_f(\varphi) = \widehat{S}(\hat{\mathbf{1}}^{(1)}) \langle \hat{f}_r, \hat{\mathbf{1}}^{(2)} \Phi^L_f(\varphi) \rightharpoonup f \rangle_F = \widehat{S}(\hat{\mathbf{1}}^{(1)}) \langle \hat{f}_r, \hat{\mathbf{1}}^{(2)} \varphi \rightharpoonup f \rangle_F$$
$$= \widehat{S}(\hat{\mathbf{1}}^{(1)}) \langle f \leftharpoonup \hat{f}_r, \hat{\mathbf{1}}^{(2)} \varphi \rangle = \widehat{\Pi}^L(\varphi^{(1)}) \langle f \leftharpoonup \hat{f}_r, \varphi^{(2)} \rangle,$$
$$\overline{\Phi}^R_f(\varphi) = \langle \hat{f}_l, \hat{\mathbf{1}}^{(1)} \overline{\Phi}^R_f(\varphi) \rightharpoonup f \rangle_F \widehat{S}^{-1}(\hat{\mathbf{1}}^{(2)}) = \langle \hat{f}_l, \hat{\mathbf{1}}^{(1)} \varphi \rightharpoonup f \rangle_F \widehat{S}^{-1}(\hat{\mathbf{1}}^{(2)})$$
$$= \langle f \leftharpoonup \hat{f}_l, \hat{\mathbf{1}}^{(1)} \varphi \rangle \widehat{S}^{-1}(\hat{\mathbf{1}}^{(2)}) = \langle f \leftharpoonup \hat{f}_l, \varphi^{(1)} \rangle \widehat{\overline{\Pi}}^R(\varphi^{(2)}) \tag{5.19}$$

using (5.17), (5.18), (1.11), and upper right and lower left equations in (1.8). Thus, using (1.7), (1.8),

$$\varphi \rightharpoonup f =: \Phi^L_f(\varphi) \rightharpoonup f = f^{(1)} \langle \widehat{\Pi}^L(\varphi^{(1)}), f^{(2)} \rangle \langle f \leftharpoonup \hat{f}_r, \varphi^{(2)} \rangle$$
$$= f^{(1)} \langle \varphi, \Pi^L(f^{(2)})(f \leftharpoonup \hat{f}_r) \rangle = \mathbf{1}^{(1)} f \langle \varphi, \mathbf{1}^{(2)}(f \leftharpoonup \hat{f}_r) \rangle, \tag{5.20a}$$
$$\varphi \rightharpoonup f =: \overline{\Phi}^R_f(\varphi) \rightharpoonup f = f^{(1)} \langle \varphi^{(1)}, f \leftharpoonup \hat{f}_l \rangle \langle \widehat{\overline{\Pi}}^R(\varphi^{(2)}), f^{(2)} \rangle$$
$$= f^{(1)} \langle \varphi, (f \leftharpoonup \hat{f}_l) \overline{\Pi}^L(f^{(2)}) \rangle = f\mathbf{1}^{(1)} \langle \varphi, (f \leftharpoonup \hat{f}_l)\mathbf{1}^{(2)} \rangle, \tag{5.20b}$$

for all $\varphi \in \widehat{A}$, which imply

$$\mathbf{1}^{(1)} f \otimes \mathbf{1}^{(2)} \big(f \leftharpoonup \hat{f}_r\big) = f^{(1)} \otimes f^{(2)} = f \mathbf{1}^{(1)} \otimes \big(f \leftharpoonup \hat{f}_l\big) \mathbf{1}^{(2)}. \tag{5.21}$$

Applying the counit ε to the first tensor factor, we obtain

$$\Pi^L(f)\big(f \leftharpoonup \hat{f}_r\big) = f = \big(f \leftharpoonup \hat{f}_l\big) \overline{\Pi}^L(f). \tag{5.22}$$

Let $g \in F$ such that $\Pi^R(g) = \mathbf{1}$. Then $g = \varphi^R \rightharpoonup f = f(\varphi^R \rightharpoonup \mathbf{1}) =: f x^R$ with $x^R \in A^R$ for some $\varphi^R \in \widehat{A}^R$ due to $\widehat{A}^R$-freeness of F and (1.6). Thus, $\mathbf{1} = \Pi^R(g) = \Pi^R(f) x^R$, that is x^R is invertible. This implies that g is also an $\widehat{A}^{L/R}$-generator of F, hence, (5.21), (5.22) hold for $f = g \in F$, too. Since $\mathbf{1} = \Pi^R(g) = S(g)(g \leftharpoonup \hat{g}_r)$ by assumption and due to the first equality of (5.21), $S(g)$, hence g, too, is invertible. Since $\overline{\Pi}^L(g) = S^{-1}(\Pi^R(g)) = \mathbf{1}$ due to (1.11), the second equality of (5.22) implies that $g \leftharpoonup \hat{g}_l = g$. Hence, the second equality of (5.21) together with invertibility of g implies that $g \in G_R(A)$ due to Corollary 5.2. The cases $g \in G_L(A), G(A)$ can be proved similarly.

(ii) First, we note that for $g, h \in G_R(A)$ $(G_L(A), G(A))$ the invertible left $\widehat{A}$-modules F_{gh} and $F_g \times F_h$ are isomorphic, because the maps

$$\begin{aligned} &U : F_g \times F_h \to F_{gh}, \quad m \otimes n \mapsto mn, \\ &V : F_{gh} \to F_g \times F_h, \quad m \mapsto \hat{\mathbf{1}}^{(1)} \rightharpoonup m h^{-1} \otimes \hat{\mathbf{1}}^{(2)} \rightharpoonup h \end{aligned} \tag{5.23}$$

are left $\widehat{A}$-module maps, which are inverses of each other. Hence, it is enough to prove that $F_g \simeq F_{\mathbf{1}}$ as left $\widehat{A}$-modules for $g \in G_R(A)$ $(G_L(A), G(A))$ iff $g \in A^T$.

Let $g \in G_R^T(A) := G_R(A) \cap A^T$. Then

$$\big(A^T\big)^{\perp} := \big\{\varphi \in \widehat{A} \,\big|\, \big\langle \varphi, A^T \big\rangle = 0\big\} \subset \widehat{A}$$

is an ideal contained in the annihilator ideal of both of the left $\widehat{A}$-modules $F_{\mathbf{1}}$ and F_g, because $F_{\mathbf{1}}, F_g \subset A^T$ and A^T is a subcoalgebra of A. Therefore, $F_{\mathbf{1}}$ and F_g are also left modules with respect to the factor algebra $\widehat{A}/(A^T)^{\perp}$ and the isomorphism of the modules $F_{\mathbf{1}}$ and F_g with respect to this factor algebra ensures their isomorphism as $\widehat{A}$-modules. The factor algebra $\widehat{A}/(A^T)^{\perp}$ is isomorphic to the dual WHA $\widehat{A^T}$ of A^T as an algebra, which is isomorphic to a direct sum of simple matrix algebras, $\widehat{A^T} \simeq \bigoplus_\alpha M_{n_\alpha}(Z_\alpha)$, due to Lemma 2.3. The Z_αs are separable field extensions of the ground field k determined by the ideal decomposition $Z = \bigoplus_\alpha Z_\alpha$ of $Z \equiv A^L \cap A^R$ and the dimensions obey $n_\alpha = \dim_{Z_\alpha} A^L_\alpha$. Hence, $F_{\mathbf{1}}$ and F_g are isomorphic $\widehat{A^T}$-modules if the multiplicities of simple submodules corresponding to the Wedderburn components of $\widehat{A^T}$ in their direct sum decompositions are equal. In order to prove this, first we note that the primitive idempotents $\{z_\alpha\}_\alpha \subset Z$ are central in A^T, hence they are in the hypercenter H of A^T and they are related to the primitive central idempotents $\{\hat{e}_\alpha\}_\alpha$ of $\widehat{A^T}$ as

$$\hat{e}_\alpha \rightharpoonup \mathbf{1} = z_\alpha = \mathbf{1} \leftharpoonup \hat{e}_\alpha \tag{5.24}$$

due to (1.6) and the remarks after it. Hence, $\hat{e}_\alpha \rightharpoonup g = (\hat{e}_\alpha \rightharpoonup \mathbf{1})g = z_\alpha g$ and $F_\mathbf{1}$ and F_g are faithful left $\widehat{A^T}$-modules, because $\mathbf{1}$ and g are invertible. Therefore, the multiplicity corresponding to a Wedderburn component of $\widehat{A^T}$ is at least one in both of the modules $F_\mathbf{1}$ and F_g. Then the identity

$$|F_\mathbf{1}| = |F_g| = \left|\widehat{A}^R\right| = \left|A^L\right| = \sum_\alpha |Z_\alpha| \dim_{Z_\alpha} A^L_\alpha = \sum_\alpha |Z_\alpha| n_\alpha \tag{5.25}$$

for k-dimensions coming from the $\widehat{A}^R$-freeness of invertible $\widehat{A}$-modules and from the algebra structure of $\widehat{A^T}$ ensures that these multiplicities are equal to one, that is $F_\mathbf{1}$ and F_g are isomorphic $\widehat{A^T} \simeq \widehat{A}/(A^T)^\perp$, hence isomorphic $\widehat{A}$-modules.

Conversely, let $g \in G_R(A)$ be such that there exists an isomorphism $U : F_\mathbf{1} \to F_g$ between the invertible left $\widehat{A}$-modules $F_\mathbf{1}$ and F_g. Using that U is an $\widehat{A}$-module map, we have

$$\begin{aligned} U(\mathbf{1})^{(1)}\big\langle\varphi, U(\mathbf{1})^{(2)}\big\rangle &= \varphi \rightharpoonup U(\mathbf{1}) = U(\varphi \rightharpoonup \mathbf{1}) = U\big(\widehat{\Pi}^L(\varphi) \rightharpoonup \mathbf{1}\big) \\ &= \widehat{\Pi}^L(\varphi) \rightharpoonup U(\mathbf{1}) = U(\mathbf{1})^{(1)}\big\langle\widehat{\Pi}^L(\varphi), U(\mathbf{1})^{(2)}\big\rangle \\ &= U(\mathbf{1})^{(1)}\big\langle\varphi, \Pi^L\big(U(\mathbf{1})^{(2)}\big)\big\rangle = \mathbf{1}^{(1)}U(\mathbf{1})\big\langle\varphi, \mathbf{1}^{(2)}\big\rangle, \quad \varphi \in \widehat{A}, \end{aligned} \tag{5.26}$$

that is $\Delta(U(\mathbf{1})) = \mathbf{1}^{(1)}U(\mathbf{1}) \otimes \mathbf{1}^{(2)}$, which ensures that $U(\mathbf{1}) \in A^L$. Moreover, $U(\mathbf{1})$ is an $\widehat{A}^{L/R}$ generator of F_g, because it is the image of the $\widehat{A}^{L/R}$ generator $\mathbf{1} \in F_\mathbf{1}$. Hence, there exists an invertible element $\varphi^L \in \widehat{A}^L$ such that

$$g = \varphi^L \rightharpoonup U(\mathbf{1}) = \big(\varphi^L \rightharpoonup \mathbf{1}\big)U(\mathbf{1}) \in A^R A^L = A^T. \tag{5.27}$$

The case of (left) grouplike elements can be proved similarly.

(iii) Let f be an $\widehat{A}^{L/R}$-generator of the invertible submodule $F_f \subset {}_{\widehat{A}}A$. If there is no right grouplike element in $F_f = \widehat{A}^L \rightharpoonup f = A^R f$, that is, due to (i), there is no such element g in F_f that obeys $\Pi^R(g) = \mathbf{1}$, let us define $g := f \leftharpoonup \hat{f}_l \in A$ with $\hat{f}_l$ given in (5.18). Then

$$F_g := \widehat{A} \rightharpoonup \big(f \leftharpoonup \hat{f}_l\big) = \big(\widehat{A} \rightharpoonup f\big) \leftharpoonup \hat{f}_l = A^R f \leftharpoonup \hat{f}_l = A^R\big(f \leftharpoonup \hat{f}_l\big) = A^R g$$

due to (1.4) and the maps

$$\leftharpoonup \hat{f}_l : F_f \to F_g, \quad x_R f \mapsto x_R f \leftharpoonup \hat{f}_l = x_R\big(f \leftharpoonup \hat{f}_l\big) = x_R g, \tag{5.28a}$$

$$\leftharpoonup\big(\overline{\Pi}^L(f) \rightharpoonup \hat{\mathbf{1}}\big) : F_g \to F_f, \quad x_R g \mapsto x_R g \leftharpoonup \big(\overline{\Pi}^L(f) \rightharpoonup \hat{\mathbf{1}}\big) = x_R g \overline{\Pi}^L(f) = x_R f, \tag{5.28b}$$

where $x_R \in A^R$, commute with the left Sweedler action, i.e., they are left $\widehat{A}$-module maps. They are also inverses of each other due to (5.22), which property has been already indicated in (5.28b). Therefore, F_g and F_f are equivalent submodules of ${}_{\widehat{A}}A$, that is F_g is also invertible. Since $\Pi^R(g) := \Pi^R(f \leftharpoonup \hat{f}_l) = \mathbf{1}$ due to (5.18) and due to the non-degeneracy of the $A^R - \widehat{A}^R$ pairing, g is a right grouplike element due to (i). The proof is similar for left grouplike elements: one has to define $g := f \leftharpoonup \hat{f}_r$ with $\hat{f}_r$ given in (5.18) to get $g \in G_L(A)$ in the submodule F_g isomorphic to F_f. □

Corollary 5.6. *The elements of $G^T_{R/L}(A) := G_{R/L}(A) \cap A^T$ are of the form $g_L S(g_L^{-1}) \in G^T_R(A)$ and $g_L S^{-1}(g_L^{-1}) \in G^T_L(A)$. They are in $G^T(A) := G(A) \cap A^T$ iff $g_L = S^2(g_L)$. $G^T_{R/L}(A)$ and $G^T(A)$ is a normal subgroup in $G_{R/L}(A)$ and $G(A)$, respectively.*

Proof. An element $g \in G^T_R(A)$ has the product form $g = g_L g_R$ due to (5.27) with $g_L := U(\mathbf{1}) \in A^L$ and $g_R := \varphi^L \rightharpoonup \mathbf{1} \in A^R$. Since g is invertible, g_L and g_R are invertible. Using property (5.2a), one obtains $\mathbf{1} = \Pi^R(g) \equiv \Pi^R(g_L g_R) = g_R S(g_L)$. The other cases follow since $G^T_L(A) = S(G^T_R(A))$ and since $G^T(A) = G^T_R(A) \cap G^T_L(A)$.

Since $\widehat{A} \rightharpoonup g \leftharpoonup \widehat{A} = gA^T = A^T g$ for $g \in G_{R/L}(A)$ $(G(A))$ due to (5.1), $gA^T g^{-1} = A^T$ follows. Therefore, $G^T_{R/L}(A)$ and $G^T(A)$ are normal subgroups. □

Proposition 5.7. *Every invertible left $\widehat{A}$-module is isomorphic to a cyclic submodule of $({}_{\widehat{A}}A, \rightharpoonup)$ generated by an element in $G_R(A)$ $(G_L(A))$. The isomorphism classes of invertible left $\widehat{A}$-modules are in one-to-one correspondence with elements of the* (*finite*) *factor group $G_R(A)/G^T_R(A)$ $(G_L(A)/G^T_L(A))$.*

Proof. Due to Proposition 5.4(ii), an invertible left $\widehat{A}$-module M is a direct sum of inequivalent simple submodules: $M = \bigoplus_p \hat{z}^L_p \cdot M =: \bigoplus_p M_p$, where $\{\hat{z}^L_p\}_p$ is the set of primitive orthogonal idempotents in $\widehat{Z}^L$. Since $\widehat{A}$ is a quasi-Frobenius algebra, see Corollary 3.3, the simple submodules M_p are isomorphic to left ideals in $\widehat{A}$ [5, p. 401]. Since they are inequivalent for different p, the invertible module ${}_{\widehat{A}}M$ itself is isomorphic to a left ideal in $\widehat{A}$. Due to Corollary 4.2, $\widehat{A}$ is a Frobenius algebra, hence, the isomorphism ${}_{\widehat{A}}\widehat{A} \simeq ({}_{\widehat{A}}A, \rightharpoonup)$ of left regular modules holds [5, p. 413]. Thus, ${}_{\widehat{A}}M$ is isomorphic to an invertible submodule of $({}_{\widehat{A}}A, \rightharpoonup)$, that is to a cyclic submodule F_g with $g \in G_R(A)$ $(g \in G_L(A))$ by Lemma 5.5(iii). Due to Lemma 5.5(ii) the isomorphism classes of cyclic submodules $F_g, g \in G_{R/L}(A)$ are given by the elements of the factor group $G_{R/L}(A)/G^T_{R/L}(A)$.

Since a finite dimensional k-algebra has a finite number of inequivalent simple modules, there is only a finite number of inequivalent semisimple modules with a given k-dimension. Therefore, the factor groups $G_{R/L}(A)/G^T_{R/L}(A)$ are finite groups. □

In consideration of Proposition 5.7, we can formulate why the notion of grouplike elements in a WHA is too restrictive: one cannot always associate a grouplike element in A to an invertible module of the dual WHA $\widehat{A}$. We formulate this claim as follows.

Proposition 5.8. *Let $t_L \in A^L_*$ denote the element that relates the counit and the reduced trace as non-degenerate functionals on the separable algebra A^L: $\varepsilon(\cdot) = \mathrm{tr}(\cdot\, t_L)$. The coset $gG^T_R(A) \subset G_R(A)$ for $g \in G_R(A)$ contains a grouplike element iff there exists $x_L \in A^L_*$ such that*

$$g t_L g^{-1} = x_L t_L x_L^{-1}. \tag{5.29}$$

In general, $G(A)/G^T(A)$ is a proper subgroup of $G_R(A)/G_R^T(A)$.

Proof. The adjoint action by $g \in G_R(A)$ on A gives rise to algebra automorphisms of A^L and A^R, because (5.1), (5.2a) imply that $\Pi^{R/L}(gy_{R/L}g^{-1}) = gy_{R/L}g^{-1}$ for $y_{R/L} \in A^{R/L}$. Using the invariance of the reduced trace with respect to algebra automorphisms and the WBA identity $\varepsilon(abc) = \varepsilon(\Pi^R(a)b\Pi^L(c))$, $a, b, c \in A$, which follows from (1.1b) and (1.3), one obtains

$$\begin{aligned}\varepsilon\big(y_L g S(g)\big) &= \varepsilon\big(\Pi^R\big(g^{-1}\big)y_L\Pi^L(g)\big) = \varepsilon\big(g^{-1}y_L g\big) = \operatorname{tr}\big(g^{-1}y_L g t_L\big)\\ &= \operatorname{tr}\big(y_L g t_L g^{-1}\big) = \varepsilon\big(y_L g t_L g^{-1} t_L^{-1}\big), \quad y_L \in A^L,\end{aligned} \tag{5.30}$$

i.e., $gS(g) = gt_L g^{-1} t_L^{-1}$ due to non-degeneracy of the counit on A^L. Therefore, for all $g \in G_R(A)$ we have

$$S(g) = t_L g^{-1} t_L^{-1}, \qquad S^2(g) = tgt^{-1}, \quad t := t_L S\big(t_L^{-1}\big). \tag{5.31}$$

The element t_L implements the Nakayama automorphism $\theta_\varepsilon = S^2$ of ε on A^L: $\theta_\varepsilon = \operatorname{Ad} t_L$. Hence, $t := t_L S(t_L^{-1}) \in A^T$ implements S^2 on A^T and due to (5.31) on the subcoalgebras gA^T of A, $g \in G_R(A)$ as well. In addition, $t \in G^T(A)$ due to Corollary 5.6.

Hence, if for a given $g \in G_R(A)$ there exists $x_L \equiv x_L(g) \in A_*^L$ such that $gt_L g^{-1} = x_L t_L x_L^{-1}$, then $gS(g) = gt_L g^{-1} t_L^{-1} = x_L t_L x_L^{-1} t_L^{-1} = x_L S^2(x_L^{-1})$ due to (5.31). Therefore, $h := x_L^{-1} S(x_L) g \in G_R(A)$ is a grouplike element in the coset $gG_R^T(A)$ because $\Pi^L(h) = \mathbf{1}$.

Conversely, if h is a grouplike element in the coset $gG_R^T(A) \subset G_R(A)$ then $h = x_L S(x_L^{-1}) g$ for some $x_L \in A_*^L$ due to Corollary 5.6. Therefore, using (5.31)

$$\mathbf{1} = \Pi^L(h) = x_L g S(g) S^2\big(x_L^{-1}\big) = x_L g t_L g^{-1} t_L^{-1} S^2\big(x_L^{-1}\big) = x_L g t_L g^{-1} x_L^{-1} t_L^{-1}. \tag{5.32}$$

For the second statement of the proposition, first we note that the inclusion $gG^T(A) \subset gG_R^T(A)$ for $g \in G(A)$ induces the inclusion $G(A)/G^T(A) \subset G_R(A)/G_R^T(A)$ of the factor groups. To show that this inclusion is proper in general an example will suffice.

Let the WHA A over the rational field $\mathbf{Q}$ be given as follows. Let A^L be a full matrix algebra $M_m(\mathbf{Q}(\sqrt{2}))$, $m > 1$, where $\mathbf{Q}(\sqrt{2})$ denotes the (separable) field extension of $\mathbf{Q}$ by $\sqrt{2}$. Let the counit ε as a non-degenerate index $\mathbf{1}$ functional on the separable algebra A^L be given with the help of the reduced trace $\varepsilon(\cdot) := \operatorname{tr}(\cdot\, t_L)$, where $t_L \in A_*^L$ satisfying $\operatorname{tr}(t_L^{-1}) = 1$. Let A^T be the WHA of the form $A^L \otimes A^{L\,\mathrm{op}} =: A^L \otimes A^R$ given in the [2, Appendix]. Let A as an algebra over $\mathbf{Q}$ be given by the crossed product $A := A^T \rtimes Z_2$, where $Z_2 = \{e, g\}$ is the cyclic group of order two and the action of the non-trivial element $g \in Z_2$ on A^L (A^R) is the outer automorphism that changes the sign of the central element $z_L = \sqrt{2} \cdot \mathbf{1}$ of A^L ($z_R = \sqrt{2} \cdot \mathbf{1} \in A^R$). Now it is a straightforward calculation that one extends the WHA structure of A^T to $A := A^T \rtimes Z_2$ by defining

$$\tilde{\varepsilon}\big(g^n x\big) := \epsilon(x), \qquad \widetilde{\Delta}\big(g^n x\big) := \big(g^n \otimes g^n\big)\Delta(x), \qquad \widetilde{S}\big(g^n x\big) := S(x) t_L g^n t_L^{-1}, \tag{5.33}$$

where $x \in A^T$ and $n = 0, 1$.

Due to Corollary 5.2, $g \in A$ becomes a right grouplike element for any possible choice of t_L, i.e., $G_R(A)/G_R^T(A) \simeq Z_2$. However, if $t_L \in A_*^L$ is such that the prescribed outer automorphism on A^L induced by g is not inner on t_L, that is (5.29) is not fulfilled, there is no grouplike element in the coset $gG_R^T(A) \subset G_R(A)$; thus, $G(A)/G^T(A) \simeq \{e\}$. □

Corollary 5.9. *$G_R(A)/G_R^T(A) = G(A)/G^T(A)$ if A^L is central simple or if $S^2|_{A^L} = \mathrm{id}|_{A^L}$. In the latter case even $G_{R/L}(A) = G(A)$ holds.*

Proof. If A^L is central simple (5.29) is fulfilled by definition. In the other case t_L is central in A^L, $G_R^T(A) = G^T(A)$ and (5.29) reads as $gt_Lg^{-1} = t_L$, $g \in G_R(A)$. Due to (5.31), $S(g)g = t_Lg^{-1}t_L^{-1}g$ and is a central element in A^L. Therefore, $\mathbf{1} = \Pi^R(g) = S(\mathbf{1}^{(1)})S(g)g\mathbf{1}^{(2)} = S(g)g$ due to (5.1), (5.2a), which proves the claim. □

6. Distinguished (left/right) grouplike elements, Radford formula, and the order of the antipode

After defining distinguished (left/right) grouplike elements and deriving some basic properties of them, we prove the generalization of the Radford formula: the fourth power of the antipode in a WHA can be expressed in terms of distinguished left (right) grouplike elements like in the finite dimensional Hopf case [15]. Using this result, we derive a finiteness type claim about the order of the antipode in a WHA and prove that the double of a WHA is unimodular.

We note that the Radford formula was proved in [13] for WHAs in the case when the square of the antipode is the identity mapping on A^L.[4] For such WHAs the sets of various grouplike elements coincide, see Corollary 5.9.

Before turning to the definition of (left/right) distinguished grouplike elements in a WHA let us examine the connection between integrals in dual pairs $A, \widehat{A}$ of WHAs.

The pair $(l, \lambda) \in I^L \times \hat{I}^L \subset A \times \widehat{A}$ ($(r, \rho) \in I^R \times \hat{I}^R$) is called a *dual pair of left* (*right*) *integrals* if they are non-degenerate and if they obey one of the equivalent relations $l \rightharpoonup \lambda = \hat{\mathbf{1}}$, $\lambda \rightharpoonup l = \mathbf{1}$ ($r \leftharpoonup \rho = \mathbf{1}$, $\rho \leftharpoonup r = \hat{\mathbf{1}}$). Due to Theorem 4.1 such pairs exist in any dual pair of WHAs. $({}_A\hat{I}^L, \star)$ is an invertible A-module due to Corollary 3.5 and Proposition 5.4(i). Since this module is the right conjugate of the module ${}_AI^R$ due to Corollary 3.4, ${}_AI^R$ is also an invertible left A-module due to (5.7), (5.8b). Hence, it is a free rank one left $A^{L/R}$-module due to Proposition 5.4(i). An element r is a free A^L (A^R) generator in ${}_AI^R$ iff r is a non-degenerate right integral, thus non-degenerate right integrals $r, r' \in I^R$ are related by an element $x_L \in A_*^L$ ($x_R \in A_*^R$): $r' = x_Lr$ ($r' = x_Rr$). The corresponding statement holds for non-degenerate right integrals in $\hat{I}^R$ by duality. Hence,

[4] For WHAs based on certain separable, but not strongly separable [9] algebra A^L the property $S^2|_{A^L} \neq \mathrm{id}|_{A^L}$, i.e., the non-triviality of the Nakayama automorphism corresponding to the counit as a non-degenerate functional $\varepsilon : A^L \to k$, is not only a possibility, but the only possibility because ε should be an index $\mathbf{1}$ functional on A^L. For example, if $A^L = M_2(\mathbf{Z}_2)$, that is a two by two matrix algebra over the finite field $\mathbf{Z}_2$, the reduced trace tr on A^L is non-degenerate but it has index 0. The two non-degenerate index $\mathbf{1}$ functional on A^L have the form $\mathrm{tr}(\cdot\, t_L)$ with $t_L^{\pm 1} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$ and lead to $S^2|_{A^L} = \mathrm{Ad}\, t_L \neq \mathrm{id}|_{A^L}$.

dual pairs of right integrals, (r_1, ρ_1) and (r_2, ρ_2), are related by a 'common' invertible element $x_L \in A_*^L$ $(x_R \in A_*^R)$:

$$(r_2, \rho_2) = \bigl(x_L r_1, \bigl(\hat{\mathbf{1}} \leftharpoonup x_L^{-1}\bigr)\rho_1\bigr) = \bigl(x_R r_1, \bigl(S^{-2}\bigl(x_R^{-1}\bigr) \rightharpoonup \hat{\mathbf{1}}\bigr)\rho_1\bigr). \tag{6.1}$$

Let us consider the element $s_R := \rho \rightharpoonup r \in A$ constructed from the elements of a dual pair (r, ρ) of right integrals. Since r is a non-degenerate functional on $\widehat{A}$ and since ρ is a free $\widehat{A}^{L/R}$-generator of the left $\widehat{A}$-module ${}_{\widehat{A}}\hat{I}^R$, s_R becomes a free left $\widehat{A}^{L/R}$-generator of the cyclic left $\widehat{A}$-module $(\widehat{A} \rightharpoonup s_R, \rightharpoonup)$, i.e., it is an invertible $\widehat{A}$-submodule in $(A, \rightharpoonup)$. Moreover, using (1.8)

$$\begin{aligned}\Pi^R(s_R) &:= \Pi^R(\rho \rightharpoonup r) = \Pi^R\bigl(r^{(1)}\bigr)\bigl\langle r^{(2)}, \rho\bigr\rangle = \mathbf{1}^{(1)}\bigl\langle r\mathbf{1}^{(2)}, \rho\bigr\rangle \\ &= \mathbf{1}^{(1)}\bigl\langle \mathbf{1}^{(2)}, \rho \leftharpoonup r\bigr\rangle = \mathbf{1}^{(1)}\bigl\langle \mathbf{1}^{(2)}, \hat{\mathbf{1}}\bigr\rangle = \mathbf{1},\end{aligned} \tag{6.2}$$

that is s_R is a right grouplike element in A due to Lemma 5.5(i). If $(r_i, \rho_i), i = 1, 2$, are dual pairs of right integrals the corresponding right grouplike elements differ by a right grouplike element in A^T due to (6.1), (1.5), (1.6), and Corollary 5.6:

$$\rho_2 \rightharpoonup r_2 = \bigl(\hat{\mathbf{1}} \leftharpoonup x_L^{-1}\bigr)\rho_1 \rightharpoonup x_L r_1 = x_L S\bigl(x_L^{-1}\bigr)(\rho_1 \rightharpoonup r_1), \quad x_L \in A_*^L. \tag{6.3}$$

However, it is not known to us whether the coset $G_R^T(A)s_R$ in $G_R(A)$ is special enough in order to contain always a grouplike element. But we note that if $s_R := \rho \rightharpoonup r$ is grouplike, i.e., $\Pi^L(s_R) = \mathbf{1}$ also holds, then $\sigma_R := r \rightharpoonup \rho \in G(\widehat{A})$ already follows: by duality σ_R is a free $A^{L/R}$-generator in the cyclic left A-module $(A \rightharpoonup \sigma_R, \rightharpoonup)$ with the property $\widehat{\Pi}^R(\sigma_R) = \hat{\mathbf{1}}$ and

$$\begin{aligned}\widehat{\Pi}^L(\sigma_R) &:= \widehat{\Pi}^L(r \rightharpoonup \rho) = \widehat{\Pi}^L\bigl(\rho^{(1)}\bigr)\bigl\langle \rho^{(2)}, r\bigr\rangle = \widehat{S}\bigl(\hat{\mathbf{1}}^{(1)}\bigr)\bigl\langle \hat{\mathbf{1}}^{(2)}\rho, r\bigr\rangle \\ &= \widehat{S}\bigl(\Pi^L(\rho \rightharpoonup r) \rightharpoonup \hat{\mathbf{1}}\bigr) =: \widehat{S}\bigl(\Pi^L(s_R) \rightharpoonup \hat{\mathbf{1}}\bigr) = \widehat{S}(\mathbf{1} \rightharpoonup \hat{\mathbf{1}}) = \hat{\mathbf{1}},\end{aligned} \tag{6.4}$$

that is σ_R is grouplike by Lemma 5.5(i).

Similarly, a dual pair (l, λ) of left integrals leads to left grouplike elements: $s_L := l \leftharpoonup \lambda \in G_L(A)$ and $\sigma_L := \lambda \leftharpoonup l \in G_L(\widehat{A})$. s_L is grouplike iff σ_L is grouplike, because $\Pi^R(s_L)$ and $\widehat{\Pi}^R(\sigma_L)$ obey a relation analogous to (6.4):

$$\Pi^R(s_L) = S\bigl(\mathbf{1} \leftharpoonup \widehat{\Pi}^R(\sigma_L)\bigr) = S(\mathbf{1} \leftharpoonup \sigma_L) = \Pi^R(\mathbf{1} \leftharpoonup \sigma_L). \tag{6.5}$$

These considerations lead to the following definition.

Definition 6.1. Let (l, λ) $((r, \rho))$ be dual pair of left (right) integrals in a dual pair $A, \widehat{A}$ of WHAs. The elements $s_L := l \leftharpoonup \lambda$ and $\sigma_L := \lambda \leftharpoonup l$ $(s_R := \rho \rightharpoonup r$ and $\sigma_R := r \rightharpoonup \rho)$ are called distinguished left (right) grouplike elements in A and $\widehat{A}$, respectively.

A dual pair of left (right) integrals is called a distinguished pair of left (right) integrals if s_L (s_R) is not only left (right) grouplike but also grouplike. In this case s_L (s_R) is called distinguished grouplike element.

Let us introduce some notations we use in the forthcoming lemma. Using properties (5.1), (5.2) it is easy to see that a left/right grouplike element $\gamma_{L/R} \in G_{L/R}(\widehat{A})$ gives rise to a projection $\Pi^{L/R}_{\gamma_{L/R}} : A \to A^{L/R}$ by defining

$$\Pi^L_{\gamma_L}(a) := \Pi^L(\gamma_L \rightharpoonup a), \qquad \Pi^R_{\gamma_R}(a) := \Pi^R(a \leftharpoonup \gamma_R), \quad a \in A. \tag{6.6}$$

The invertible right/left A-module structures of left/right integrals in A can be made explicit by using these projections and distinguished left/right grouplike elements $\sigma_{L/R}$ connected to the dual pair $(l,\lambda)/(r,\rho)$ of left/right integrals

$$la = l\Pi^R_{\widehat{S}(\sigma_L^{-1})}(a), \qquad ar = \Pi^L_{\widehat{S}(\sigma_R^{-1})}(a)r, \quad a \in A. \tag{6.7}$$

For example, the first relation can be proved by using (5.1), (5.2b), (1.6), and the non-degeneracy of λ:

$$\begin{aligned}\lambda \leftharpoonup la &= \sigma_L \leftharpoonup a = \big\langle \widehat{S}\big(\sigma_L^{-1}\big)\hat{\mathbf{1}}^{(1)}, a\big\rangle \sigma_L \hat{\mathbf{1}}^{(2)} = \sigma_L\big(\hat{\mathbf{1}} \leftharpoonup \Pi^R\big(a \leftharpoonup \widehat{S}\big(\sigma_L^{-1}\big)\big)\big)\\ &= \sigma_L \leftharpoonup \Pi^R\big(a \leftharpoonup \widehat{S}\big(\sigma_L^{-1}\big)\big) = \lambda \leftharpoonup l\Pi^R_{\widehat{S}(\sigma_L^{-1})}(a).\end{aligned}$$

Lemma 6.2. *Let $C_b := AbA \subset A$ be the cyclic ideal with the generator $b = b(\gamma,\delta) \in A$ characterized by a left and a right grouplike element $\gamma \in G_L(\widehat{A})$ and $\delta \in G_R(\widehat{A})$, respectively, through the property*

$$abc = \Pi^L_\gamma(a)b\Pi^R_\delta(c), \quad a, c \in A, \tag{6.8}$$

where the projections Π^L_γ and Π^R_δ are defined in (6.6). *The left/right Sweedler actions by left/right grouplike elements in $\widehat{A}$ provide isomorphisms between such types of cyclic ideals as* (*possibly non-unital*) *rings. The image $\tilde{b}$ of the generator $b = b(\gamma,\delta)$ obeys the characterization property*

$$\beta_L \rightharpoonup b \leftharpoonup \beta_R =: \tilde{b} = \tilde{b}\big(\widehat{S}(\beta_R)\gamma\beta_L^{-1}, \beta_R^{-1}\delta\widehat{S}(\beta_L)\big), \quad \beta_{L/R} \in G_{L/R}(\widehat{A}). \tag{6.9}$$

Proof. First, we note that the set of such cyclic ideals is non-empty: $l \in I^L$ from a dual pair (l,λ) of left integrals is a generator with characterization property $l = l(\hat{\mathbf{1}}, \widehat{S}(\sigma_L^{-1}))$ due to (1.9) and (6.7), where $\sigma_L := l \leftharpoonup \lambda$ is the corresponding distinguished left grouplike element.

Since left (right) Sweedler actions by left (right) grouplike elements in $\widehat{A}$ provide algebra automorphisms of A, the isomorphism of the corresponding cyclic ideals as rings follows. The only open question is the characterization property (6.9) of the image $\tilde{b}$ of the generator $b = b(\gamma,\delta)$. Using properties (6.1), (6.2b) of left grouplike elements, characterization property (6.8) of the generator b, coproduct properties (1.4) of elements in $A^{L/R}$, and properties (1.7) of the projections $\Pi^{L/R}$ and $\widehat{\Pi}^{L/R}$, one derives

$$\begin{aligned} a(\beta_L \rightharpoonup b) &= \beta_L \rightharpoonup \big(\beta_L^{-1} \rightharpoonup a\big)b = \beta_L \rightharpoonup \Pi^L_\gamma\big(\beta_L^{-1} \rightharpoonup a\big)b = \Pi^L_\gamma\big(\beta_L^{-1} \rightharpoonup a\big)(\beta_L \rightharpoonup b) \\ &=: \Pi^L\big(\gamma \rightharpoonup \big(\beta_L^{-1} \rightharpoonup a\big)\big)(\beta_L \rightharpoonup b) = \Pi^L_{\gamma\beta_L^{-1}}(a)(\beta_L \rightharpoonup b), \end{aligned} \tag{6.10a}$$

$$\begin{aligned} (\beta_L \rightharpoonup b)c &= \beta_L \rightharpoonup b\big(\beta_L^{-1} \rightharpoonup c\big) = \beta_L \rightharpoonup b\Pi^R_\delta\big(\beta_L^{-1} \rightharpoonup c\big) \\ &= (\beta_L \rightharpoonup b)\big(\beta_L \rightharpoonup \Pi^R_\delta\big(\beta_L^{-1} \rightharpoonup c\big)\big) = (\beta_L \rightharpoonup b)\mathbf{1}^{(1)}\big\langle \beta_L, \Pi^R_\delta\big(\beta_L^{-1} \rightharpoonup c\big)\mathbf{1}^{(2)}\big\rangle \\ &= (\beta_L \rightharpoonup b)\mathbf{1}^{(1)}\big\langle \hat{\mathbf{1}}^{(1)}\beta_L \otimes \hat{\mathbf{1}}^{(2)}\beta_L, \Pi^R\big(\beta_L^{-1} \rightharpoonup c \leftharpoonup \delta\big) \otimes \mathbf{1}^{(2)}\big\rangle \\ &= (\beta_L \rightharpoonup b)\mathbf{1}^{(1)}\big\langle \widehat{\Pi}^R\big(\hat{\mathbf{1}}^{(1)}\beta_L\big) \otimes \widehat{\Pi}^L\big(\hat{\mathbf{1}}^{(2)}\beta_L\big), \beta_L^{-1} \rightharpoonup c \leftharpoonup \delta \otimes \mathbf{1}^{(2)}\big\rangle \\ &= (\beta_L \rightharpoonup b)\mathbf{1}^{(1)}\big\langle \widehat{S}(\beta_L)\hat{\mathbf{1}}^{(1)}\beta_L \otimes \hat{\mathbf{1}}^{(2)}, \beta_L^{-1} \rightharpoonup c \leftharpoonup \delta \otimes \mathbf{1}^{(2)}\big\rangle \\ &= (\beta_L \rightharpoonup b)\mathbf{1}^{(1)}\big\langle \Delta(\hat{\mathbf{1}}), \big(c \leftharpoonup \delta\widehat{S}(\beta_L)\big) \otimes \mathbf{1}^{(2)}\big\rangle \\ &= (\beta_L \rightharpoonup b)\mathbf{1}^{(1)}\varepsilon\big(\big(c \leftharpoonup \delta\widehat{S}(\beta_L)\big)\mathbf{1}^{(2)}\big) \\ &= (\beta_L \rightharpoonup b)\Pi^R\big(c \leftharpoonup \delta\widehat{S}(\beta_L)\big) = (\beta_L \rightharpoonup b)\Pi^R_{\delta\widehat{S}(\beta_L)}(c). \end{aligned} \tag{6.10b}$$

The change of the characterization property of the generator b due to right Sweedler actions $b \leftharpoonup \beta_R$, $\beta_R \in G_R(\widehat{A})$ can be proved similarly. □

Corollary 6.3. *Distinguished left grouplike elements in $\widehat{A}$ fall into a central element of the factor group $G_L(\widehat{A})/G_L^T(\widehat{A})$. There exists a two-sided non-degenerate integral in A iff distinguished left grouplike elements in $\widehat{A}$ fall into the unit element of this factor group.*

Proof. For any $\beta \in G_L(\widehat{A})$ the map $B_\beta(a) := \beta \rightharpoonup a \leftharpoonup \widehat{S}^{-1}(\beta)$, $a \in A$ defines an algebra automorphism of A, which maps the space I^L of left integrals into itself due to the previous lemma. The image $\tilde{l} := B_\beta(l)$ of a non-degenerate left integral $l = l(\hat{\mathbf{1}}, \widehat{S}(\sigma_L^{-1}))$ is a non-degenerate left integral having the characterization property $\tilde{l} = \tilde{l}(\hat{\mathbf{1}}, \widehat{S}^{-1}(\beta^{-1})\widehat{S}(\sigma_L^{-1})\widehat{S}(\beta))$ due to (6.9). Hence, the distinguished left grouplike element $\tilde{\sigma}_L$ corresponding to $\tilde{l}$ is given by

$$\begin{aligned} &\tilde{\sigma}_L = \widehat{S}^{-2}(\beta)\sigma_L\beta^{-1} =: \varphi\beta\sigma_L\beta^{-1}, \\ &S^{-2}(\beta)\beta^{-1} =: \varphi = \widehat{S}^{-1}\big(\varphi_L^{-1}\big)\varphi_L \in G_L^T(\widehat{A}), \end{aligned} \tag{6.11}$$

with $\varphi_L = \widehat{S}^{-1}(\widehat{\Pi}^R(\beta^{-1})) \in \widehat{A}_*^L$ due to the form (5.2b) of $\widehat{\Pi}^R(\beta^{-1})$. However, distinguished left grouplike elements differ by elements in $G_L^T(\widehat{A})$, in analogy with the case (6.3) of distinguished right grouplike elements. Hence, for the $G_L^T(\widehat{A})$-cosets (6.11) implies the relation $[\sigma_L] = [\tilde{\sigma}_L] = [\varphi][\beta][\sigma_L][\beta]^{-1} = [\beta][\sigma_L][\beta]^{-1}$, that is $[\sigma_L]$ is central in the factor group $G_L(\widehat{A})/G_L^T(\widehat{A})$.

If the non-degenerate left integral $l \in I^L$ is also a right integral then we have the relation $\Pi^R_{\widehat{S}(\sigma_L^{-1})} = \Pi^R$ due to (6.7) and (1.9). Hence, $\sigma_L = \hat{\mathbf{1}}$ since

$$\langle \hat{\mathbf{1}}, a\rangle = \big\langle \hat{\mathbf{1}}, \Pi^R(a)\big\rangle = \big\langle \hat{\mathbf{1}}, \Pi^R_{\widehat{S}(\sigma_L^{-1})}(a)\big\rangle = \big\langle \hat{\mathbf{1}}, a \leftharpoonup \widehat{S}\big(\sigma_L^{-1}\big)\big\rangle = \big\langle \widehat{S}\big(\sigma_L^{-1}\big), a\big\rangle, \quad a \in A,$$

using (6.6) and (1.7). Conversely, if $[\sigma_L]$ is the unit element of the factor group then there exists a dual pair (l, λ) of left integrals with distinguished left grouplike element $\sigma_L = \hat{\mathbf{1}}$ due to a relation analogous with (6.3). Therefore, $\Pi^R_{\widehat{S}(\sigma_L^{-1})} = \Pi^R$ and (6.7) implies that l is a (non-degenerate) two-sided integral. $\square$

Theorem 6.4. *Let $A, \widehat{A}$ be a dual pair of WHAs and let (s_L, σ_L) be the pair of distinguished left grouplike elements corresponding to a dual pair (l, λ) of left integrals in $A \times \widehat{A}$. The Nakayama automorphism $\theta_\lambda := \widehat{R}_\lambda^{-1} \circ \widehat{L}_\lambda : A \to A$ corresponding to the non-degenerate functional $\lambda : A \to k$ can be written as*

$$\theta_\lambda(a) = \sigma_L^{-1} \rightharpoonup S^2(a) = s_L^{-1} S^{-2}(a) s_L \leftharpoonup \widehat{S}^{-1}(\sigma_L), \quad a \in A. \tag{6.12}$$

The fourth power of the antipode S of A can be written as

$$S^4(a) = \sigma_L \rightharpoonup s_L^{-1} a s_L \leftharpoonup \widehat{S}^{-1}(\sigma_L), \quad a \in A. \tag{6.13}$$

The order of the antipode is finite up to an inner automorphism by a grouplike element in the trivial subalgebra A^T.

Proof. In the sufficiency proof of Theorem 4.1 we have seen that the antipode and its inverse can be given with the help of pairs of non-degenerate integrals $l/r \in I^{L/R}$, $\lambda/\rho \in \hat{I}^{L/R}$

$$S(a) = \big(R_l \circ \widehat{L}_\lambda\big)(a) := (\lambda \leftharpoonup a) \rightharpoonup l, \quad \lambda \rightharpoonup l = \mathbf{1},\ l \rightharpoonup \lambda = \hat{\mathbf{1}}, \tag{6.14a}$$

$$S^{-1}(a) = \big(L_l \circ \widehat{L}_\rho\big)(a) := l \leftharpoonup (\rho \leftharpoonup a), \quad l \leftharpoonup \rho = \mathbf{1},\ l \rightharpoonup \rho = \hat{\mathbf{1}}, \tag{6.14b}$$

$$S^{-1}(a) = \big(R_r \circ \widehat{R}_\lambda\big)(a) := (a \rightharpoonup \lambda) \rightharpoonup r, \quad \lambda \leftharpoonup r = \hat{\mathbf{1}},\ \lambda \rightharpoonup r = \mathbf{1}, \tag{6.14c}$$

$$S(a) = \big(L_r \circ \widehat{R}_\rho\big)(a) := r \leftharpoonup (a \rightharpoonup \rho), \quad r \leftharpoonup \rho = \mathbf{1},\ \rho \leftharpoonup r = \hat{\mathbf{1}}. \tag{6.14d}$$

Choosing a dual pair (l, λ) of left integrals, we rewrite the antipode relations (6.14b–d) in terms of (l, λ) and the corresponding pair $(s, \sigma) \equiv (s_L, \sigma_L)$ of distinguished left grouplike elements. We note that the second relations between the members of integral pairs given in (6.14a–d) are consequences of the first ones (see the proof of Theorem 4.1), hence, it is enough to ensure only these ones.

For (6.14b) the new member of the required pair (l, ρ) of integrals is given by $\rho := \widehat{S}^{-1}(\lambda) = (\lambda \leftharpoonup s)\widehat{\Pi}^R(\sigma)^{-1}$. Indeed, ρ is a non-degenerate right integral and $\lambda = \widehat{S}(\rho) = (l \leftharpoonup \rho) \rightharpoonup \lambda$ implies the relation $l \leftharpoonup \rho = \mathbf{1}$ due to injectivity of $\widehat{R}_\lambda$. Moreover, using property (1.16) of left integrals

$$\begin{aligned}(\lambda \leftharpoonup s)\widehat{\Pi}^R(\sigma)^{-1} &:= \big(\lambda \leftharpoonup (l \leftharpoonup \lambda)\big)\widehat{\Pi}^R(\sigma)^{-1} = \big\langle \lambda\lambda^{(1)}, l\big\rangle \lambda^{(2)}\widehat{\Pi}^R(\sigma)^{-1} \\ &= \big\langle \lambda^{(1)}, l\big\rangle \widehat{S}^{-1}(\lambda)\lambda^{(2)}\widehat{\Pi}^R(\sigma)^{-1} = \widehat{S}^{-1}(\lambda)\sigma\widehat{\Pi}^R(\sigma)^{-1} \\ &= \widehat{S}^{-1}(\lambda)\widehat{\Pi}^R(\sigma)\widehat{\Pi}^R(\sigma)^{-1} = \widehat{S}^{-1}(\lambda) = \rho.\end{aligned} \tag{6.15}$$

Hence, interchanging the role of A and $\widehat{A}$, the new member of integrals for (6.14c) is given by $r := S^{-1}(l) = (l \leftharpoonup \sigma)\Pi^R(s)^{-1}$. For (6.14d) the pair is given by $(r := S^{-1}(l), \rho := \widehat{S}(\lambda) = s \rightharpoonup \lambda)$, because $\rho = \widehat{S}(\lambda) = (l \leftharpoonup \lambda) \rightharpoonup \lambda = s \rightharpoonup \lambda$ and $r = S^{-1}(l)$ are non-degenerate right integrals and $r \leftharpoonup \rho = S^{-1}(l) \leftharpoonup \widehat{S}(\lambda) = S^{-1}(\lambda \rightharpoonup l) = \mathbf{1}$. Therefore, we can rewrite (6.14b and c) as

$$\begin{aligned} S^{-1}(a) &= l \leftharpoonup (\rho \leftharpoonup a) = l \leftharpoonup \big((\lambda \leftharpoonup s)\widehat{\Pi}^R(\sigma)^{-1} \leftharpoonup a\big) = l \leftharpoonup (\lambda \leftharpoonup sa)\widehat{\Pi}^R(\sigma)^{-1} \\ &= \big[l \leftharpoonup (\lambda \leftharpoonup sa)\big]\big(\mathbf{1} \leftharpoonup \widehat{\Pi}^R(\sigma)^{-1}\big) = \big(L_l \circ \widehat{L}_\lambda\big)(sa)\big(\mathbf{1} \leftharpoonup \widehat{\Pi}^R(\sigma)^{-1}\big), \qquad (6.16b) \\ S^{-1}(\sigma \rightharpoonup a) &= S^{-1}(a) \leftharpoonup \widehat{S}(\sigma) = \big((a \rightharpoonup \lambda) \rightharpoonup r\big) \leftharpoonup \widehat{S}(\sigma) \\ &= (a \rightharpoonup \lambda) \rightharpoonup (l \leftharpoonup \sigma)\Pi^R(s)^{-1} \leftharpoonup \widehat{S}(\sigma) \\ &= (a \rightharpoonup \lambda) \rightharpoonup \big(l \leftharpoonup \sigma\widehat{S}(\sigma)\big)\Pi^R(s)^{-1} \\ &= (a \rightharpoonup \lambda) \rightharpoonup \big(l \leftharpoonup \widehat{\Pi}^R\big(\sigma^{-1}\big)^{-1}\big)\Pi^R(s)^{-1} \\ &= (a \rightharpoonup \lambda) \rightharpoonup l\big(\mathbf{1} \leftharpoonup \widehat{\Pi}^R\big(\sigma^{-1}\big)^{-1}\big)\Pi^R(s)^{-1} \\ &= (a \rightharpoonup \lambda) \rightharpoonup l\Pi^R\big(\big(\mathbf{1} \leftharpoonup \sigma\widehat{S}(\sigma)\big) \leftharpoonup \widehat{S}\big(\sigma^{-1}\big)\big)\Pi^R(s)^{-1} \\ &= (a \rightharpoonup \lambda) \rightharpoonup l\Pi^R(\mathbf{1} \leftharpoonup \sigma)\Pi^R(s)^{-1} = (a \rightharpoonup \lambda) \rightharpoonup l\Pi^R(s)\Pi^R(s)^{-1} \\ &= (a \rightharpoonup \lambda) \rightharpoonup l = (R_l \circ \widehat{R}_\lambda)(a), \qquad (6.16c) \end{aligned}$$

using relations (1.4) and (1.6) for elements in A^R, the identity $\sigma\widehat{S}(\sigma) = \widehat{\Pi}^R(\sigma^{-1})^{-1}$ following from (5.2b), the right A-module property (6.7) of left integrals, and the relation (6.5). Finally, using property (1.16) of left integrals, (6.14d) can be rewritten as

$$\begin{aligned} S(a) &= r \leftharpoonup (a \rightharpoonup \rho) = (l \leftharpoonup \sigma)\Pi^R(s)^{-1} \leftharpoonup \big(a \rightharpoonup (s \rightharpoonup \lambda)\big) \\ &= \big[(l \leftharpoonup \sigma) \leftharpoonup (as \rightharpoonup \lambda)\big]\Pi^R(s)^{-1} = \big[l \leftharpoonup \sigma\lambda^{(1)}\langle\lambda^{(2)}, as\rangle\big]\Pi^R(s)^{-1} \\ &= \big[l \leftharpoonup \lambda^{(1)}\langle\widehat{S}^{-1}(\sigma)\lambda^{(2)}, as\rangle\big]\Pi^R(s)^{-1} = \big[l \leftharpoonup \big(\big(as \leftharpoonup \widehat{S}^{-1}(\sigma)\big) \rightharpoonup \lambda\big)\big]\Pi^R(s)^{-1} \\ &= (L_l \circ \widehat{R}_\lambda)\big(as \leftharpoonup \widehat{S}^{-1}(\sigma)\big)\Pi^R(s)^{-1}. \qquad (6.16d) \end{aligned}$$

Therefore, using (6.14a), (6.16b–d), the algebra isomorphism property of the map $\hat{\kappa}_R$ given in (1.5), the relation (6.5), and the form (5.2b) of $\Pi^R(s)$, we get

$$\begin{aligned} \big(R_l \circ \widehat{L}_\lambda\big)(a) &= S(a) = S^{-1}\big(\sigma \rightharpoonup \big(\sigma^{-1} \rightharpoonup S^2(a)\big)\big) \\ &= \big(R_l \circ \widehat{R}_\lambda\big)\big(\sigma^{-1} \rightharpoonup S^2(a)\big), \qquad (6.17a) \\ \big(L_l \circ \widehat{L}_\lambda\big)(a) &= S^{-1}\big(s^{-1}a\big)\big(\mathbf{1} \leftharpoonup \widehat{\Pi}^R(\sigma)\big) = S^{-1}\big(\Pi^R(s)s^{-1}a\big) = S^{-1}(a)s \\ &= S\big[s^{-1}S^{-2}(a)\big]\Pi^R(s) = \big(L_l \circ \widehat{R}_\lambda\big)\big(s^{-1}S^{-2}(a)s \leftharpoonup \widehat{S}^{-1}(\sigma)\big). \qquad (6.17b) \end{aligned}$$

Due to injectivity of R_l and L_l (6.17a and b) lead to connections between $\widehat{R}_\lambda$ and $\widehat{L}_\lambda$ that imply (6.12). The equality of these two different forms of the Nakayama automorphism θ_λ gives rise to the Radford formula (6.13).

Since left (right) Sweedler actions by left (right) grouplike elements are algebra automorphisms, iterating the Radford formula m times, one arrives at

$$S^{4m}(a) = S^{4m}(s^{-1}) \dots S^4(s^{-1})\bigl(\sigma^m \rightharpoonup a \leftharpoonup \widehat{S}^{-1}(\sigma^m)\bigr) S^4(s) \dots S^{4m}(s), \quad a \in A. \tag{6.18}$$

For $g \in G_L(A)$ the relation $S^2(g) = S(\Pi^R(g^{-1})^{-1})\Pi^R(g^{-1})g \in G_L^T(A)g$ holds due to (5.2b) and Corollary 5.6. Hence, $S^{2n}(g) \in G_L^T(A)g$ is for any integer n. Since the factor group $G_L(A)/G_L^T(A)$ is finite due to Proposition 5.7, there exists an integer m and $x \equiv S(x_R)x_R^{-1} \in G_L^T(A)$, $\varphi \equiv \widehat{S}(\varphi_R)\varphi_R^{-1} \in G_L^T(\widehat{A})$ with $x_R \in A_*^R$, $\varphi_R \in \widehat{A}_*^R$ such that (6.18) reads as

$$\begin{aligned}
S^{4m}(a) &= x^{-1}\bigl(\varphi \rightharpoonup a \leftharpoonup \widehat{S}^{-1}(\varphi)\bigr)x \\
&= x^{-1}\bigl(\widehat{S}(\varphi_R) \rightharpoonup \mathbf{1}\bigr)\bigl(\mathbf{1} \leftharpoonup \widehat{S}^{-1}(\varphi_R^{-1})\bigr)a\bigl(\varphi_R^{-1} \rightharpoonup \mathbf{1}\bigr)(\mathbf{1} \leftharpoonup \varphi_R)x \\
&= x^{-1}S^{-1}(\mathbf{1} \leftharpoonup \varphi_R)\bigl(\mathbf{1} \leftharpoonup \varphi_R^{-1}\bigr)aS^{-1}\bigl(\mathbf{1} \leftharpoonup \varphi_R^{-1}\bigr)(\mathbf{1} \leftharpoonup \varphi_R)x \\
&= x^{-1}S^{-1}(\mathbf{1} \leftharpoonup \varphi_R)(\mathbf{1} \leftharpoonup \varphi_R)^{-1}aS^{-1}(\mathbf{1} \leftharpoonup \varphi_R)^{-1}(\mathbf{1} \leftharpoonup \varphi_R)x \\
&= S\bigl(y_R^{-1}\bigr)y_R a S(y_R)y_R^{-1}, \quad a \in A,
\end{aligned} \tag{6.19}$$

where we used the identities (1.6) and the notation $y_R := x_R S^{-1}(\mathbf{1} \leftharpoonup \varphi_R) \in A_*^R$. Due to (6.19), S^{4m} is an inner algebra automorphism of A by an element $y := S(y_R)y_R^{-1} \in G_L^T(A)$. However, S^{4m} is also a coalgebra automorphism of A, which requires y to be a grouplike element. Indeed, using the coproduct property (1.4) and separability identities (1.12) for A^L and A^R, one derives the relation

$$\begin{aligned}
\Delta(a) &= \bigl(S(y_R) \otimes y_R^{-1}\bigr)\Delta\bigl(S\bigl(y_R^{-1}\bigr)y_R a S(y_R)y_R^{-1}\bigr)\bigl(S\bigl(y_R^{-1}\bigr) \otimes y_R\bigr) \\
&= \bigl(S(y_R) \otimes y_R^{-1}\bigr)\Delta\bigl(S^{4m}(a)\bigr)\bigl(S\bigl(y_R^{-1}\bigr) \otimes y_R\bigr) \\
&= \bigl(S(y_R) \otimes y_R^{-1}\bigr)\bigl(S^{4m} \otimes S^{4m}\bigr)\bigl(\Delta(a)\bigr)\bigl(S\bigl(y_R^{-1}\bigr) \otimes y_R\bigr) \\
&= \bigl(y_R \otimes S\bigl(y_R^{-1}\bigr)\bigr)\Delta(a)\bigl(y_R^{-1} \otimes S(y_R)\bigr) = \bigl(y_R S^2\bigl(y_R^{-1}\bigr) \otimes \mathbf{1}\bigr)\Delta(a)\bigl(y_R y_R^{-1} \otimes \mathbf{1}\bigr) \\
&= \bigl(y_R S^2\bigl(y_R^{-1}\bigr) \otimes \mathbf{1}\bigr)\Delta(a), \quad a \in A,
\end{aligned} \tag{6.20}$$

which leads to the equality $\mathbf{1} = y_R S^2(y_R^{-1})$ by applying the counit to the second tensor factor. Hence, $y = S(y_R)y_R^{-1}$ is not only in $G_L^T(A)$ but also in $G^T(A)$ due to Corollary 5.6, which together with (6.19) proves the last claim in the theorem. □

The Radford formula [15] was used in [16] to prove unimodularity of the Drinfeld double $\mathcal{D}(H)$ of a Hopf algebra H. In the case of the double $\mathcal{D}(A)$ of a WHA A [3] the same result holds:

Corollary 6.5. *The double $\mathcal{D}(A)$ of a WHA A is unimodular, i.e., there exists a non-degenerate two-sided integral in $\mathcal{D}(A)$. Namely, if (l, λ) is a dual pair of left integrals in $A \times \widehat{A}$ then $\mathcal{D}(l \otimes \widehat{S}(\lambda))$ is a two-sided non-degenerate integral in $\mathcal{D}(A)$.*

Proof. The double $\mathcal{D}(A)$ of a WHA A [3] is the k-linear space of the tensor product of A and $\widehat{A}$ over the subalgebras $A^L \simeq \widehat{A}^R$ and $A^R \simeq \widehat{A}^L$

$$\begin{aligned}&\mathcal{D}(A) \ni \mathcal{D}(ax_Lx_R \otimes \varphi) = \mathcal{D}\big(a \otimes \big(x_L \rightharpoonup \hat{\mathbf{1}}\big)\big(\hat{\mathbf{1}} \leftharpoonup x_R\big)\varphi\big),\\ &a \in A,\ \varphi \in \widehat{A},\ x_{L/R} \in A^{L/R}\end{aligned} \tag{6.21}$$

together with the WHA structure maps

$$\begin{aligned}\mathcal{D}(a \otimes \varphi)\mathcal{D}(b \otimes \psi) &:= \mathcal{D}\big(ab^{(2)} \otimes \varphi^{(2)}\psi\big)\big\langle a^{(1)}, \widehat{S}^{-1}\big(\varphi^{(3)}\big)\big\rangle\big\langle a^{(3)}, \varphi^{(1)}\big\rangle,\\ \varepsilon_{\mathcal{D}}\big(\mathcal{D}(a \otimes \varphi)\big) &:= \varepsilon\big(a(\varphi \rightharpoonup \mathbf{1})\big) = \hat{\varepsilon}\big(\big(\hat{\mathbf{1}} \leftharpoonup a\big)\varphi\big),\\ \Delta_{\mathcal{D}}\big(\mathcal{D}(a \otimes \varphi)\big) &:= \mathcal{D}\big(a^{(1)} \otimes \varphi^{(2)}\big) \otimes \mathcal{D}\big(a^{(2)} \otimes \varphi^{(1)}\big),\\ S_{\mathcal{D}}\big(\mathcal{D}(a \otimes \varphi)\big) &:= \mathcal{D}\big(\mathbf{1} \otimes \widehat{S}^{-1}(\varphi)\big)\mathcal{D}\big(S(a) \otimes \hat{\mathbf{1}}\big).\end{aligned} \tag{6.22}$$

Let (l, λ) be a dual pair of left integrals in $A \times \widehat{A}$ with the corresponding pair $(s, \sigma) \equiv (s_L, \sigma_L)$ of distinguished left grouplike elements. The expression (6.12) of the Nakayama automorphism θ_l corresponding to l implies that $\Delta^{op}(l) = l^{(1)} \otimes S^2(l^{(2)}s_L^{-1})$. Hence, using properties (5.1), (5.2b) of a left grouplike element

$$\begin{aligned}l^{(2)} \otimes l^{(3)}s^{-1}S^{-1}\big(l^{(1)}\big) &= l^{(1)} \otimes l^{(2)}s^{-1}S\big(l^{(3)}s^{-1}\big)\\ &= l^{(1)} \otimes l^{(2)}\mathbf{1}^{(1)}s^{-1}S\big(s^{-1}\big)S\big(\mathbf{1}^{(2)}\big)S\big(l^{(3)}\big)\\ &= l^{(1)} \otimes l^{(2)}\Pi^L\big(s^{-1}\big)S\big(l^{(3)}\big) = l^{(1)} \otimes \Pi^L\big(l^{(2)}\big).\end{aligned} \tag{6.23}$$

Moreover, the first relation in (6.7) implies that for $\lambda \in \hat{I}^L$ and $\varphi \in \widehat{A}$

$$\begin{aligned}\varphi\widehat{S}(\lambda) &= \widehat{S}\big(\lambda\widehat{\Pi}^R_{S(s^{-1})}\big(\widehat{S}^{-1}(\varphi)\big)\big) = \big(\widehat{S} \circ \widehat{\Pi}^R \circ \widehat{S}^{-1}\big)\big(s^{-1} \rightharpoonup \varphi\big)\widehat{S}(\lambda)\\ &= \widehat{\Pi}^L_{s^{-1}}(\varphi)\widehat{S}(\lambda),\end{aligned} \tag{6.24}$$

where we used (1.10) in the third equality. The relations (6.23), (6.24) and the properties (6.21), (6.22) of $\mathcal{D}(A)$ together with the identities (1.6) and (1.11) lead to

$$\begin{aligned}&\mathcal{D}(a \otimes \varphi)\mathcal{D}\big(l \otimes \widehat{S}(\lambda)\big)\\ &\quad := \mathcal{D}\big(al^{(2)} \otimes \varphi^{(2)}\widehat{S}(\lambda)\big)\big\langle l^{(1)}, \widehat{S}^{-1}(\varphi^{(3)})\big\rangle\big\langle l^{(3)}, \varphi^{(1)}\big\rangle\\ &\quad = \mathcal{D}\big(al^{(2)} \otimes \widehat{\Pi}^L\big(s^{-1} \rightharpoonup \varphi^{(2)}\big)\widehat{S}(\lambda)\big)\big\langle S^{-1}\big(l^{(1)}\big), \varphi^{(3)}\big\rangle\big\langle l^{(3)}, \varphi^{(1)}\big\rangle\\ &\quad = \mathcal{D}\big(al^{(2)}\big(\widehat{\Pi}^L\big(\varphi^{(2)}\big) \rightharpoonup \mathbf{1}\big) \otimes \widehat{S}(\lambda)\big)\big\langle l^{(3)}, \varphi^{(1)}\big\rangle\big\langle s^{-1}S^{-1}\big(l^{(1)}\big), \varphi^{(3)}\big\rangle\\ &\quad = \mathcal{D}\big(a\big(\widehat{\widehat{\Pi}}^R\big(\varphi^{(2)}\big) \rightharpoonup l^{(2)}\big) \otimes \widehat{S}(\lambda)\big)\big\langle l^{(3)}, \varphi^{(1)}\big\rangle\big\langle s^{-1}S^{-1}\big(l^{(1)}\big), \varphi^{(3)}\big\rangle\\ &\quad = \mathcal{D}\big(al^{(2)} \otimes \widehat{S}(\lambda)\big)\big\langle l^{(3)}, \widehat{\widehat{\Pi}}^R\big(\varphi^{(2)}\big)\varphi^{(1)}\big\rangle\big\langle s^{-1}S^{-1}\big(l^{(1)}\big), \varphi^{(3)}\big\rangle\\ &\quad = \mathcal{D}\big(al^{(2)} \otimes \widehat{S}(\lambda)\big)\big\langle l^{(3)}s^{-1}S^{-1}\big(l^{(1)}\big), \varphi\big\rangle\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{D}\big(al^{(1)} \otimes \widehat{S}(\lambda)\big)\big\langle \Pi^L\big(l^{(2)}\big), \varphi\big\rangle = \mathcal{D}\big(a\big(\widehat{\Pi}^L(\varphi) \rightharpoonup l\big) \otimes \widehat{S}(\lambda)\big) \\
&= \mathcal{D}\big(aS\big(\widehat{\Pi}^L(\varphi) \rightharpoonup \mathbf{1}\big)l \otimes \widehat{S}(\lambda)\big) = \mathcal{D}\big(a\big(\mathbf{1} \leftharpoonup \widehat{S}^{-1}\big(\widehat{\Pi}^L(\varphi)\big)\big)l \otimes \widehat{S}(\lambda)\big) \\
&= \mathcal{D}\big(\Pi^L\big(a\big(\mathbf{1} \leftharpoonup \widehat{\widehat{\Pi}}^R(\varphi)\big)\big)l \otimes \widehat{S}(\lambda)\big) = \mathcal{D}\big(\Pi^L\big(a\big(\mathbf{1} \leftharpoonup \widehat{\widehat{\Pi}}^R(\varphi)\big)\big) \otimes \hat{\mathbf{1}}\big)\mathcal{D}\big(l \otimes \widehat{S}(\lambda)\big) \\
&= \Pi^L_{\mathcal{D}}\big(\mathcal{D}(a \otimes \varphi)\big)\mathcal{D}\big(l \otimes \widehat{S}(\lambda)\big), \quad a \in A, \varphi \in \widehat{A},
\end{aligned} \tag{6.25}
$$

that is $\mathcal{D}(l \otimes \widehat{S}(\lambda))$ is a left integral in $\mathcal{D}(A)$. A similar computation shows that it is also a right integral.

Now, we prove that $\mathcal{D}(l \otimes \widehat{S}(\lambda))$ is a non-degenerate functional on the dual $\widehat{\mathcal{D}}(A)$ of $\mathcal{D}(A)$. The WHA $\widehat{\mathcal{D}}(A)$ [3] is the k-linear space of the tensor product of $\widehat{A}$ and A over the subalgebras $\widehat{A}^R \simeq A^L$ and $\widehat{A}^L \simeq A^R$

$$
\widehat{\mathcal{D}}(A) \ni \widehat{\mathcal{D}}\big(\varphi \otimes x_L a S^{-1}(x_R)\big) = \widehat{\mathcal{D}}\big(\widehat{S}^{-1}\big(\hat{\mathbf{1}} \leftharpoonup x_R\big)\varphi\big(x_L \rightharpoonup \hat{\mathbf{1}}\big) \otimes a\big), \tag{6.26}
$$

where $\varphi \in \widehat{A}$, $a \in A$, $x_{L/R} \in A^{L/R}$. The WHA structure maps of $\widehat{\mathcal{D}}(A)$ are transposed to that of $\mathcal{D}(A)$ with respect to the non-degenerate pairing

$$
\begin{aligned}
&\big\langle \widehat{\mathcal{D}}(\varphi \otimes a), \mathcal{D}(b \otimes \psi)\big\rangle := \big\langle \varphi \otimes a, P(b \otimes \psi)\big\rangle = \big\langle \widehat{P}(\varphi \otimes a), b \otimes \psi\big\rangle, \\
&a, b \in A,\ \varphi, \psi \in \widehat{A},
\end{aligned} \tag{6.27}
$$

where $P : A \otimes \widehat{A} \to A \otimes \widehat{A}$ and $\widehat{P} : \widehat{A} \otimes A \to \widehat{A} \otimes A$ are k-linear projections given with the help of separating idempotents of A^L and A^R

$$
\begin{aligned}
P(b \otimes \psi) &:= b\mathbf{1}^{(1)}S\big(\mathbf{1}^{(1')}\big) \otimes \big(\mathbf{1}^{(2')} \rightharpoonup \hat{\mathbf{1}}\big)\big(\hat{\mathbf{1}} \leftharpoonup S\big(\mathbf{1}^{(2)}\big)\big)\psi, \\
\widehat{P}(\varphi \otimes a) &:= \big(\mathbf{1}^{(1')} \rightharpoonup \hat{\mathbf{1}}\big)\varphi\big(\mathbf{1}^{(1)} \rightharpoonup \hat{\mathbf{1}}\big) \otimes \mathbf{1}^{(2)}a\mathbf{1}^{(2')}.
\end{aligned} \tag{6.28}
$$

Clearly, $P(A \otimes \widehat{A})$ and $\mathcal{D}(A)$ ($\widehat{P}(\widehat{A} \otimes A)$ and $\widehat{\mathcal{D}}(A)$) are isomorphic k-linear spaces and $\mathcal{D}(P(b \otimes \psi)) = \mathcal{D}(b \otimes \psi)$ ($\widehat{\mathcal{D}}(\widehat{P}(\varphi \otimes a)) = \widehat{\mathcal{D}}(\varphi \otimes a)$) also holds due to (6.21) (or (6.26)). Since $\mathcal{D}(A)$ is finite dimensional, the two-sided integral $\mathcal{D}(l \otimes \widehat{S}(\lambda))$ is non-degenerate if the k-linear map $R_{\mathcal{D}(l \otimes \widehat{S}(\lambda))} : \widehat{\mathcal{D}}(A) \to \mathcal{D}(A)$ is injective, that is if $0 = \widehat{\mathcal{D}}(\varphi \otimes a) \rightharpoonup \mathcal{D}(l \otimes \widehat{S}(\lambda))$ implies $0 = \widehat{\mathcal{D}}(\varphi \otimes a)$. Using the mentioned isomorphisms of the k-linear spaces, the definition (6.28) of the projection P, the form of the coproduct in $\mathcal{D}(A)$, and the identity

$$
\big(\hat{\mathbf{1}} \leftharpoonup \Pi^R_{\widehat{S}(\sigma^{-1})}\big(S\big(\mathbf{1}^{(1)}\big)\big)\big)\widehat{\Pi}^L_{s^{-1}}\big(\mathbf{1}^{(2)} \rightharpoonup \hat{\mathbf{1}}\big) = \hat{\mathbf{1}}
$$

we prove later on, one computes

$$
\begin{aligned}
&P\big(l^{(1)} \otimes \widehat{S}(\lambda)^{(2)}\big)\big\langle \widehat{\mathcal{D}}(\varphi \otimes a), \mathcal{D}\big(l^{(2)} \otimes \widehat{S}(\lambda)^{(1)}\big)\big\rangle \\
&\quad := l^{(1)}\mathbf{1}^{(1)}S\big(\mathbf{1}^{(1')}\big) \otimes \big(\mathbf{1}^{(2')} \rightharpoonup \hat{\mathbf{1}}\big)\big(\hat{\mathbf{1}} \leftharpoonup S\big(\mathbf{1}^{(2)}\big)\big)\widehat{S}(\lambda)^{(2)}\big\langle \widehat{\mathcal{D}}(\varphi \otimes a), \mathcal{D}\big(l^{(2)} \otimes \widehat{S}(\lambda)^{(1)}\big)\big\rangle
\end{aligned}
$$

$$
\begin{aligned}
&= l^{(1)} S\big(\mathbf{1}^{(1')}\big) \otimes \big(\mathbf{1}^{(2')} \rightharpoonup \hat{\mathbf{1}}\big) \widehat{S}(\lambda)^{(2)} \big\langle \widehat{\mathcal{D}}(\varphi \otimes a), \\
&\qquad\qquad\qquad \mathcal{D}\big(l^{(2)} S\big(\mathbf{1}^{(1)}\big) \otimes \widehat{S}\big(\hat{\mathbf{1}} \leftharpoonup S\big(\mathbf{1}^{(2)}\big)\big) \widehat{S}(\lambda)^{(1)}\big)\big\rangle \\
&= l^{(1)} S\big(\mathbf{1}^{(1')}\big) \otimes \big(\mathbf{1}^{(2')} \rightharpoonup \hat{\mathbf{1}}\big) \widehat{S}(\lambda)^{(2)} \big\langle \widehat{\mathcal{D}}(\varphi \otimes a), \mathcal{D}\big(l^{(2)} S\big(\mathbf{1}^{(1)}\big)\mathbf{1}^{(2)} \otimes \widehat{S}(\lambda)^{(1)}\big)\big\rangle \\
&= \big[l S\big(\mathbf{1}^{(1)}\big)\big]^{(1)} \otimes \big[\big(\mathbf{1}^{(2)} \rightharpoonup \hat{\mathbf{1}}\big) \widehat{S}(\lambda)\big]^{(2)} \big\langle \widehat{\mathcal{D}}(\varphi \otimes a), \\
&\qquad\qquad\qquad \mathcal{D}\big(\big[l S\big(\mathbf{1}^{(1)}\big)\big]^{(2)} \otimes \big[\big(\mathbf{1}^{(2)} \rightharpoonup \hat{\mathbf{1}}\big) \widehat{S}(\lambda)\big]^{(1)}\big)\big\rangle \\
&= l^{(1)} \otimes \widehat{S}(\lambda)^{(2)} \big\langle \widehat{\mathcal{D}}(\varphi \otimes a), \mathcal{D}\big(l^{(2)} \Pi^R_{\widehat{S}(\sigma^{-1})}\big(S\big(\mathbf{1}^{(1)}\big)\big) \otimes \widehat{\Pi}^L_{s^{-1}}\big(\mathbf{1}^{(2)} \rightharpoonup \hat{\mathbf{1}}\big) \widehat{S}(\lambda)^{(1)}\big)\big\rangle \\
&= l^{(1)} \otimes \widehat{S}(\lambda)^{(2)} \big\langle \widehat{\mathcal{D}}(\varphi \otimes a), \mathcal{D}\big(l^{(2)} \otimes \big(\hat{\mathbf{1}} \leftharpoonup \Pi^R_{\widehat{S}(\sigma^{-1})}\big(S\big(\mathbf{1}^{(1)}\big)\big)\big) \widehat{\Pi}^L_{s^{-1}}\big(\mathbf{1}^{(2)} \rightharpoonup \hat{\mathbf{1}}\big) \widehat{S}(\lambda)^{(1)}\big)\big\rangle \\
&= l^{(1)} \otimes \widehat{S}(\lambda)^{(2)} \big\langle \widehat{\mathcal{D}}(\varphi \otimes a), \mathcal{D}\big(l^{(2)} \otimes \widehat{S}(\lambda)^{(1)}\big)\big\rangle \\
&= l^{(1)} \otimes \widehat{S}(\lambda)^{(2)} \big\langle \widehat{P}(\varphi \otimes a), l^{(2)} \otimes \widehat{S}(\lambda)^{(1)}\big\rangle = \big(R_l \otimes \widehat{L}_{\widehat{S}(\lambda)}\big)\big(\widehat{P}(\varphi \otimes a)\big),
\end{aligned}
\tag{6.29}
$$

where we used (1.16) in the second equality, (1.4) in the fourth, and (6.7) and (1.10) in the fifth one. The k-linear map $R_l \otimes \widehat{L}_{\widehat{S}(\lambda)} : \widehat{A} \otimes A \to A \otimes \widehat{A}$ is injective due to the non-degeneracy of the integrals l and λ. Hence, (6.29) implies that $\widehat{P}(\varphi \otimes a)$, or equivalently $\widehat{\mathcal{D}}(\varphi \otimes a)$, should be zero if the left-hand side of (6.29), or equivalently $\widehat{\mathcal{D}}(\varphi \otimes a) \rightharpoonup \mathcal{D}(l \otimes \widehat{S}(\lambda))$, is zero. Finally, the proof of the identity we used in (6.29) is as follows:

$$
\begin{aligned}
&\big(\hat{\mathbf{1}} \leftharpoonup \Pi^R_{\widehat{S}(\sigma^{-1})}\big(S\big(\mathbf{1}^{(1)}\big)\big)\big) \widehat{\Pi}^L_{s^{-1}}\big(\mathbf{1}^{(2)} \rightharpoonup \hat{\mathbf{1}}\big) \\
&\quad = \big(\hat{\mathbf{1}} \leftharpoonup \Pi^R\big(S\big(\mathbf{1}^{(1)}\big) \leftharpoonup \widehat{S}\big(\sigma^{-1}\big)\big)\big) \widehat{\Pi}^L\big(s^{-1}\mathbf{1}^{(2)} \rightharpoonup \hat{\mathbf{1}}\big) \\
&\quad = \big(\hat{\mathbf{1}} \leftharpoonup S\big(S\big(\mathbf{1}^{(1)}\big) \leftharpoonup \widehat{S}\big(\sigma^{-1}\big)\big)\big) \widehat{S}\big(\hat{\mathbf{1}}^{(1)}\big) \big\langle s^{-1}\mathbf{1}^{(2)}, \hat{\mathbf{1}}^{(2)}\big\rangle \\
&\quad = \widehat{S}\big(\hat{\mathbf{1}}^{(1)}\big) \big\langle s^{-1}\mathbf{1}^{(2)}, \hat{\mathbf{1}}^{(2)}\big(\hat{\mathbf{1}} \leftharpoonup \big(\sigma^{-1} \rightharpoonup S^2\big(\mathbf{1}^{(1)}\big)\big)\big)\big\rangle \\
&\quad = \widehat{S}\big(\hat{\mathbf{1}}^{(1)}\big) \big\langle s^{-1}\mathbf{1}^{(2)}, \hat{\mathbf{1}}^{(2)} \leftharpoonup \big(\sigma^{-1} \rightharpoonup S^2\big(\mathbf{1}^{(1)}\big)\big)\big\rangle \\
&\quad = \widehat{S}\big(\hat{\mathbf{1}}^{(1)}\big) \big\langle \big(\sigma^{-1} \rightharpoonup S^2\big(\mathbf{1}^{(1)}\big)\big) s^{-1}\mathbf{1}^{(2)}, \hat{\mathbf{1}}^{(2)}\big\rangle \\
&\quad = \widehat{S}\big(\hat{\mathbf{1}}^{(1)}\big) \big\langle S^2\big(\mathbf{1}^{(1)}\big)\big(\sigma \rightharpoonup s^{-1}\mathbf{1}^{(2)}\big), \hat{\mathbf{1}}^{(2)}\sigma^{-1}\big\rangle \\
&\quad = \widehat{S}\big(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma\big) \big\langle S^2\big(\mathbf{1}^{(1)}\big)\big(\sigma \rightharpoonup s^{-1}\mathbf{1}^{(2)}\big), \widehat{S}^{-1}(\sigma)\hat{\mathbf{1}}^{(2)}\big\rangle \\
&\quad = \widehat{S}\big(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma\big) \big\langle S^2\big(\mathbf{1}^{(1)}\big)\big(\sigma \rightharpoonup s^{-1}\mathbf{1}^{(2)}\big) \leftharpoonup \widehat{S}^{-1}(\sigma), \hat{\mathbf{1}}^{(2)}\big\rangle \\
&\quad = \widehat{S}\big(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma\big) \big\langle S^2\big(\mathbf{1}^{(1)}\big)\big(\sigma \rightharpoonup s^{-1}\mathbf{1}^{(2)}ss^{-1} \leftharpoonup \widehat{S}^{-1}(\sigma)\big), \hat{\mathbf{1}}^{(2)}\big\rangle \\
&\quad = \widehat{S}\big(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma\big) \big\langle S^2\big(\mathbf{1}^{(1)}\big) S^4\big(\mathbf{1}^{(2)}\big) S^4\big(s^{-1}\big), \hat{\mathbf{1}}^{(2)}\big\rangle \\
&\quad = \widehat{S}\big(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma\big) \big\langle \Pi^L\big(S^2\big(\mathbf{1}^{(1)}\big) S^4\big(\mathbf{1}^{(2)}\big) S^4\big(s^{-1}\big)\big), \hat{\mathbf{1}}^{(2)}\big\rangle \\
&\quad = \widehat{S}\big(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma\big) \big\langle S^4\big(\mathbf{1}^{(2)}\big) S^3\big(\mathbf{1}^{(1)}\big), \hat{\mathbf{1}}^{(2)}\big\rangle = \widehat{S}\big(\sigma^{-1}\hat{\mathbf{1}}^{(1)}\sigma\big) \big\langle \mathbf{1}, \hat{\mathbf{1}}^{(2)}\big\rangle = \hat{\mathbf{1}},
\end{aligned}
$$

where we used (1.10) in the second equality, (1.12) and the anticoalgebra map property of the antipode in the third one, algebra automorphism properties of Sweedler actions by left grouplike elements in the sixth, (5.1), (5.2b) in the seventh and twelfth, the Radford formula in the tenth, and (1.7) in the eleventh equality. □

Appendix A

Here we give examples of finite dimensional WHAs of $A = A^L \otimes A^R \equiv B \otimes B^{\mathrm{op}}$ type, where B is a separable k-algebra equipped with a non-degenerate functional $E : B \to k$ of index $\mathbf{1}$ (see [2, Appendix]), having antipode of infinite order.

Let $B = M_n(\mathbf{R})$, i.e., a full matrix algebra over the real field, and let $\mathrm{tr} : B \to \mathbf{R}$ denote the trace functional with $\mathrm{tr}(\mathbf{1}) = n$. Any invertible element $t \in B$ with $\mathrm{tr}(t^{-1}) = 1$ defines a non-degenerate functional $E : B \to \mathbf{R}$ by

$$E(x) := \mathrm{tr}(tx), \quad x \in B, \tag{A.1}$$

which has index $\mathbf{1}$. Indeed, if $\{e_{ab}\}$ is a set of matrix units then $\{e_i\}_i \equiv \{t^{-1}e_{ab}\}_{(a,b)}$ and $\{f_i\}_i \equiv \{e_{ba}\}_{(a,b)}$ are dual $\mathbf{R}$-bases of B with respect to E, $E(e_i f_j) = \delta_{ij}$, and the index of E is

$$\mathrm{Ind}\, E := \sum_i f_i e_i = \sum_{a,b} e_{ba} t^{-1} e_{ab} = \mathrm{tr}\big(t^{-1}\big) \sum_b e_{bb} = \mathbf{1}. \tag{A.2}$$

The Nakayama automorphism θ of E defined by $E(xy) =: E(y\theta(x))$, $x, y \in B$ is inner,

$$\theta(x) = txt^{-1}, \quad x \in B \tag{A.3}$$

due to the form (A.1) of E. We can construct the WHA $B \otimes B^{\mathrm{op}}$ [2]: it is the $\mathbf{R}$-linear space $B \otimes B$ with structure maps

$$(x_1 \otimes x_2)(y_1 \otimes y_2) := x_1 y_1 \otimes y_2 x_2, \qquad \Delta(x_1 \otimes x_2) := \sum_i (x_1 \otimes f_i) \otimes (e_i \otimes x_2),$$
$$\varepsilon(x_1 \otimes x_2) := E(x_1 x_2), \qquad S(x_1 \otimes x_2) := x_2 \otimes \theta(x_1). \tag{A.4}$$

Clearly, $S^2 = \theta \otimes \theta$, therefore the form (A.3) of the Nakayama automorphism θ shows that the order of the antipode S is finite iff $t^m \in \mathrm{Center}\, B$ for a certain positive integer m. However, this is not the case for a generic invertible $t \in B = M_n(\mathbf{R})$ with $\mathrm{tr}(t^{-1})$.

Although the order of the antipode is not finite in the generic case, already S^2 is an inner automorphism by a grouplike element in the trivial subalgebra A^T, which, in this case, is equal to A itself. Indeed,

$$S^2(x \otimes y) = \theta(x) \otimes \theta(y) = \big(t \otimes t^{-1}\big)(x \otimes y)\big(t^{-1} \otimes t\big), \tag{A.5}$$

and $t \otimes t^{-1} = (t \otimes \mathbf{1})S(t^{-1} \otimes \mathbf{1})$ with $t \otimes \mathbf{1} = S^2(t \otimes \mathbf{1}) \in A^L$. Therefore $t \otimes t^{-1}$ is a grouplike element in the trivial subalgebra A^T by Corollary 5.6.

Appendix B

Here we give the generalization of the cyclic module [4] $A^{\natural}_{(\sigma,s)}$ for weak Hopf algebras having a modular pair (σ, s) in involution. The details will be published elsewhere.

Let A be a weak Hopf algebra. The pair $(\sigma, s) \in G(\widehat{A}) \times G(A)$ of grouplike elements is called a *modular pair* for A if

$$\sigma \rightharpoonup s = s = s \leftharpoonup \sigma, \qquad s \rightharpoonup \sigma = \sigma = \sigma \leftharpoonup s. \tag{B.1}$$

They form a *modular pair in involution* if they implement the square of the antipode

$$S^2(a) = \sigma \rightharpoonup sas^{-1} \leftharpoonup \sigma^{-1}, \quad a \in A. \tag{B.2}$$

Clearly, a modular pair (in involution) is a self-dual notion for WHAs.

The identity (B.2) is a kind of square root of the Radford formula, hence, modular pairs in involution do not exist for arbitrary WHAs. However, there is a wide class of WHAs having such a pair. For example, in a weak Hopf C^*-algebra A there is a canonical grouplike element $g \in A$ implementing S^2 on A [2], hence $(\hat{\mathbf{1}}, g)$ is a modular pair in involution for A. Another example is as follows: let A be a WHA over k and let the WHA $A_G := \langle A^T, G_R(A)\rangle$ be the subWHA of A generated by the trivial subWHA A^T and by (a subgroup of) the right grouplike elements $G_R(A)$ in A. Then $(\hat{\mathbf{1}}, t)$ with $t \in G^T(A)$ defined in (5.31) is a modular pair in involution for A_G, because t implements S^2 for A^T and $G_R(A)$ due to (5.31).

Proposition. *Let A be a WHA over the field k and $(\sigma, s) \in G(\widehat{A}) \times G(A)$ be a modular pair in involution. Let the cochains $C^n_{(\sigma,s)}(A)$, $n \geqslant 0$ be defined by the n-fold product of the left regular module ${}_AA$, i.e., the k-linear spaces*

$$\begin{aligned} C^0_{(\sigma,s)}(A) &:= A^L, \\ C^n_{(\sigma,s)}(A) &:= A \times A \times \cdots \times A \equiv \Delta^{n-1}(\mathbf{1}) \cdot (A \otimes A \otimes \cdots \otimes A). \end{aligned} \tag{B.3}$$

The face operators $\delta_i^{(n)} : C^{n-1}_{(\sigma,s)}(A) \to C^n_{(\sigma,s)}(A)$, $0 \leqslant i \leqslant n$ are

$$\begin{aligned} \delta_0^{(1)}(x_L) &:= \overline{\Pi}^R(x_L), \\ \delta_1^{(1)}(x_L) &:= x_L s, \\ \delta_0^{(n)}(a_1 \otimes \cdots \otimes a_{n-1}) &:= \mathbf{1}^{(1)} \otimes \mathbf{1}^{(2)} a_1 \otimes a_2 \otimes \cdots \otimes a_{n-1}, \quad 1 < n, \\ \delta_i^{(n)}(a_1 \otimes \cdots \otimes a_{n-1}) &:= a_1 \otimes a_2 \otimes \cdots \otimes \Delta(a_i) \otimes \cdots \otimes a_{n-1}, \quad 1 \leqslant i < n,\ 1 < n, \\ \delta_n^{(n)}(a_1 \otimes \cdots \otimes a_{n-1}) &:= a_1 \otimes \cdots \otimes a_{n-2} \otimes \mathbf{1}^{(1)} a_{n-1} \otimes \mathbf{1}^{(2)} s, \quad 1 < n, \end{aligned} \tag{B.4}$$

the degeneracy operators $\sigma_i^{(n)} : C^{n+1}_{(\sigma,s)}(A) \to C^n_{(\sigma,s)}(A)$, $0 \leqslant i \leqslant n$ are

$$\sigma_0^{(0)}(a) := \Pi^L(a),$$

$$\sigma_i^{(n)}(a_1 \otimes \cdots \otimes a_{n+1}) := a_1 \otimes \cdots \otimes \Pi^L(a_{i+1})a_{i+2} \otimes \cdots \otimes a_{n+1}, \quad 0 \leqslant i < n,\ 0 < n,$$
$$\sigma_n^{(n)}(a_1 \otimes \cdots \otimes a_{n+1}) := a_1 \otimes \cdots \otimes a_{n-1} \otimes \overline{\Pi}^R(a_{n+1})a_n, \quad 0 < n, \tag{B.5}$$

and the cyclic operators $\tau_{(n)} : C^n_{(\sigma,s)}(A) \to C^n_{(\sigma,s)}(A)$ *are given by*

$$\tau_{(0)}(x_L) := x_L,$$
$$\tau_{(n)}(a_1 \otimes \cdots \otimes a_n) := \Delta^{(n-1)}\big(S(a_1 \leftharpoondown \sigma)\big) \cdot (a_2 \otimes \cdots \otimes a_n \otimes s), \quad n \geqslant 1. \tag{B.6}$$

With the definitions (B.3)–(B.6) $A^{\natural}_{(\sigma,s)} \equiv \{C^n_{(\sigma,s)}(A)\}_{n\geqslant 0}$ *becomes a* Λ*-module, where* Λ *is the cyclic category.*

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